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Orthogonal Functions

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ORTHOGONAL FUNCTIONS

being

A thesis presented to the Graduate Faculty
of the Fort Hays Kansas State College in
partial fulfillment of the requirements for
the Degree of Master of Science

by

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It was the purpose of this thesis (1) to investigate certain known orthogonal functions; (2) to exhibit elements of similarity among them; (3) to show that they arise as the solutions of differential equations in a similar way; (4) to establish that the differential equations which yield them are similar and are special cases of yet another more general differential equation; (5) to verify that the Sturm-Liouville theory can be applied to each of these differential equations; and (6) to demonstrate how each of these orthogonal functions is used to give a series representation of a given function.

By direct computation it was shown that the orthogonal functions studied arose from their respective differential equations in a similar way. These differential equations were then shown to be special cases of a more general differential equation, thus showing another similarity among the orthogonal functions. The Sturm-Liouville theory was used to find certain information in order to establish the orthogonality of the functions that were a solution of the various differential equations. Finally, by direct methods each of the orthogonal functions was utilized in forming a series representation of two arbitrary functions and these series representations were put in graphic form to show yet another similarity of the orthogonal functions that were studied.

ACKNOWLEDGEMENTS

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INTRODUCTION

Within the last several decades the differential equation as a tool to solve more complex problems has become more and more important. Its application is unlimited in the field of electronic and mechanical computers, to say nothing of its continued use in the solutions of problems in mathematics and physics. It is then important that these various differential equations can be solved and from their solution the necessary data be obtained. The use of orthogonal functions is a particular method of solution.

Definition of orthogonal functions.[1] The functions F(x) and G(x) are said to be orthogonal to one another on the interval (a,b) if

$$\int_a^b F(x)\,G(x)\,dx = 0. \tag{I-1}$$

The choice of the word orthogonal is made because of an analogy between the condition (I-1) and the condition that two vectors A and B be orthogonal. If A and B have components A_1, A_2, A_3, and B_1, B_2, B_3, respectively, the vectors are orthogonal if and only if their dot product

$$A \cdot B = A_1B_1 + A_2B_2 + A_3B_3 = 0. \tag{I-2}$$

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[1] Angus E. Taylor, Advanced Calculus, Boston, 1955, pp. 722-723.

The analogy between (I-1) and (I-2) is seen more clearly if the integral in (I-1) is expressed as a limit of a sum:

$$\lim_{n \to \infty} \left[F(x_1)\ G(x_1) + F(x_2)\ G(x_2) + \cdots + F(x_n)\ G(x_n) \right] \frac{b-a}{n} = 0,$$

where $x_k = a + \frac{k}{n}(b - a)$, $k = 1, 2, \ldots, n$. This analogy may at first seem rather farfetched. It has real substance, however, and is just one instance of a very remarkable and useful crossing over of ideas from the geometry of vector spaces to the study of functions. In many parts of mathematics it is useful to regard a function as an object much like a vector and a whole class of functions as a mathematical system much like a vector space.

If two functions $F(x)$ and $G(x)$ do not satisfy condition (I-1), they can be orthogonal to each other with respect to a weight function $p(x)$ on the interval (a,b). The condition (I-1) is changed to

(I-3) $$\int_a^b p(x)\ F(x)\ G(x)\ dx = 0.$$

In other words, $F(x)$ and $G(x)$ are multiplied by some function $p(x)$ in order to make them orthogonal to each other over the interval (a,b). If the weight function is $p(x) = 1$, condition (I-1) is obtained and is, therefore, a special case of (I-3).

Statement of the problem. It was the purpose of this thesis (1) to investigate certain known orthogonal functions; (2) to exhibit elements of similarity among them; (3) to show that they arise as the solutions of differential equations in a similar way; (4) to establish that the differential equations which yield them are similar and are special cases of yet another more general differential equation; (5) to verify that the Sturm-Liouville theory can be applied to each of these differential equations; and (6) to demonstrate how each of these orthogonal functions is used to give a series representation of a given function.

Limitations of the problem. The orthogonal functions investigated in this thesis were limited to the following:

1. Fourier
2. Hermite
3. Jacobi
4. Legendre
5. Tchebicheff
6. Laguerre
7. Bessel.

The given functions that are represented by a series through the use of orthogonal functions are limited to two, namely: (1) $f(x) = 0$ in the interval to the left of the origin and $f(x) = 1$ in the interval to the right of the origin and (2) $f(x) = -x$ in the interval to the left of the origin and $f(x) = x$ in the interval to the right of the origin. However, since the Laguerre and Bessel functions have their interval of orthogonality only to the right of the origin, the given functions were limited to (1) $f(x) = 1$, and (2) $f(x) = x$, over their respective intervals.

CHAPTER II

THE FOURIER SERIES

<u>Differential equation.</u> The differential equation associated with the Fourier series is[1]:

$$y'' + n^2 y = 0,$$

where $y = A_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx); \qquad -\pi < x < \pi.$

<u>General differential equation.</u> Investigation of the applicability of the Pearson differential equation[2],

$$(A + Bx + Cx^2)\, y'' + \left[(B + D) + (2C + E)\, x\right] y' - \left[C\, n(n + 1) + E\, n\right] y = 0,$$

to the Fourier differential equation immediately reveals that $A = 1$; $B = C = 0$ and the lack of a coefficient of the y' term eliminate the possibility of having a coefficient of the y term in the Pearson differential equation, since $C = E = 0$. Hence, it is evident that the Pearson differential equation, while it does include as special cases the differential equations associated with several of the orthogonal functions studied, does not hold for the Fourier differential equation.

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[1]Dunham Jackson, <u>Fourier Series and Orthogonal Polynomials</u>, Buffalo, 1941, p. 88.

[2]<u>Ibid.</u>, p. 164.

Further investigation of a more general differential equation of the form:

$$(A_1 + B_1x + C_1x^2)\ y'' + (D_1 + E_1x + F_1x^2)\ y' + (G_1 + H_1x + J_1x^2)\ y = 0,$$

and the substitution $A_1 = 1$, $G_1 = n^2$, $B_1 = C_1 = D_1 = E_1 = F_1 = H_1 = J_1 = 0$, produces the Fourier differential equation. Furthermore, if $A_1 = A$, $B_1 = B$, $C_1 = C$, $D_1 = (B + D)$, $E_1 = (2C + E)$, $F_1 = H_1 = J_1 = 0$, $G_1 = -\left[C\ n(n + 1) + E\ n\right]$, the Pearson differential equation can be considered a special case of this more general differential equation.

A special case of the above differential equation can be determined that will not only satisfy the Fourier differential equation but also the Hermite differential equation, which will be taken up in the next chapter. This special case can be written as the Fourier-Hermite differential equation:

$$y'' + Ax\ y' + B\ y = 0.$$

The Fourier differential equation will be obtained if $A = 0$ and $B = n^2$. It will be shown in the next chapter how the Hermite differential equation is obtained from the Fourier-Hermite differential equation.

Sturm-Liouville theory. The Sturm-Liouville theory states[3] that when applied to a more general partial differential equation, separation of variables will yield an equation in $X(x)$ of the type

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[3] Ruel V. Churchill, Fourier Series and Boundary Value Problems, New York, 1941, pp. 47-50.

$$X'' + f_1(x)\, X' + \left[f_2(x) + \lambda f_3(x)\right] X = 0.$$

Here f_1, f_2, and f_3 are known functions involved in the coefficients of the partial differential equation, and λ is the constant which arises upon separation of variables.

When the last equation is multiplied through by the factor $r(x)$, where

$$r(x) = e^{\int f_1(x)\, dx},$$

it takes the form

$$\text{(II-1)} \qquad \frac{d}{dx}\left[r(x)\frac{dX}{dx}\right] + \left[q(x) + \lambda\, p(x)\right] X = 0,$$

known as the more general Sturm-Liouville equation.

The boundary conditions on $X(x)$ have the form

$$\text{(II-2)} \qquad a_1\, X(a) + a_2\, X'(a) = 0, \qquad b_1\, X(b) + b_2\, X'(b) = 0,$$

where a_1, a_2, b_1, and b_2 are constants.

Furthermore, in case $r(a) = 0$, the first of the conditions (II-2) can be dropped from the problem, and if $r(b) = 0$, the second of those conditions can be dropped. If $r(a) = r(b)$, those conditions can be replaced by the periodic conditions,

$$\text{(II-3)} \qquad X(a) = X(b), \qquad X'(a) = X'(b).$$

Let the coefficients p, q, and r in the Sturm-Liouville problem be continuous in the interval $a \leqslant x \leqslant b$, and let λ_m, λ_n be any two distinct characteristic numbers and $X_m(x)$, $X_n(x)$ be the corresponding characteristic functions, whose derivatives $X_m'(x)$, $X_n'(x)$ are continuous. Then $X_m(x)$ and $X_n(x)$ are orthogonal on the interval (a,b), with respect to the weight function $p(x)$.

The validity of the last statement in the preceding paragraph can readily be shown. Since X_m and X_n are solutions of equation (II-1) when $\lambda = \lambda_m$ and $\lambda = \lambda_n$, respectively,

$$\frac{d}{dx}(r X_m') + (q + \lambda_m p) X_m = 0,$$

$$\frac{d}{dx}(r X_n') + (q + \lambda_n p) X_n = 0.$$

Multiplication of the first equation by X_n and the second by X_m, and subtraction give

$$(\lambda_m - \lambda_n)\, p\, X_m X_n = X_m \frac{d}{dx}(r X_n') - X_n \frac{d}{dx}(r X_m')$$

$$= \frac{d}{dx}\left[(r X_n') X_m - (r X_m') X_n\right].$$

By integration of both members over the interval (a,b),

(II-4) $$(\lambda_m - \lambda_n)\int_a^b p\, X_m X_n\, dx = \Big[r\,(X_m X_n' - X_n X_m')\Big]_a^b .$$

If the right-hand member vanishes because of the conditions set down in (II-2) and $(\lambda_m - \lambda_n) \neq 0$, then

(II-5) $$\int_a^b p(x)\, X_m(x)\, X_n(x)\, dx = 0,$$

which is the statement of orthogonality between X_m and X_n.

Application of this theory to the Fourier differential equation:

$$y'' + n^2 y = 0,$$

discloses that $f_1(x) = 0$,

so, $$r(x) = e^{\int f_1(x)\,dx} = e^{\int (0)\,dx} = e^{0} = 1.$$

The Fourier differential equation can be written in the form

$$\frac{d}{dx}\left[(1)\ \frac{dy}{dx}\right] + n^2\ y = 0,$$

from which

$$q(x) = 0, \qquad \lambda = n^2, \qquad \text{and} \quad p(x) = 1.$$

If $a = -\pi$, and $b = \pi$, in condition (II-3), then

(II-6) $$X(-\pi) = X(\pi), \qquad X'(-\pi) = X'(\pi),$$

since $r(x) = 1$, then $r(a) = r(b)$. Substitution of $a = -\pi$, $b = \pi$, $p(x) = 1$, and $r(x) = 1$, in equation (II-4)

$$(\lambda_m - \lambda_n)\int_{-\pi}^{\pi}(1)\ X_m(x)\ X_n(x)\ dx = \Big[(1)\Big\{X_m(x)\ X_n'(x) - X_n(x)\ X_m'(x)\Big\}\Big]_{-\pi}^{\pi}$$

$$= X_m(\pi)\ X_n'(\pi) - X_n(\pi)\ X_m'(\pi)$$
$$- X_m(-\pi)\ X_n'(-\pi) + X_n(-\pi)\ X_m'(-\pi)$$
$$= X_m(\pi)\ X_n'(\pi) - X_n(\pi)\ X_m'(\pi)$$
$$- X_m(\pi)\ X_n'(\pi) + X_n(\pi)\ X_m'(\pi) = 0.$$

(II-7) $$\int_{-\pi}^{\pi} X_m(x)\ X_n(x)\ dx = 0.$$

Since $\lambda_m - \lambda_n \neq 0$, $X_m(x)$ and $X_n(x)$ are orthogonal with respect to each other.

It is now left to determine the $X(x)$'s in condition (II-3).

From (II-6) $\qquad X(-\pi) = X(\pi), \qquad X'(-\pi) = X'(\pi).$

If $\qquad X_m(x) = a_m \cos mx,$

then $\qquad a_m \cos -m\pi = a_m \cos m\pi, \quad -a_m m \sin -m\pi = -a_m m \sin m\pi.$

These results are known to be true from elementary trigonometry.

If $\qquad X_n(x) = b_n \sin nx,$

then $b_n \sin -n\pi = b_n \sin n\pi$, $b_n n \cos -n\pi = b_n n \cos n\pi$.

These results again are known to be valid.

Replacement of the X's in (II-7) by cosines and sines gives:

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0, \quad \text{where } m,n = 0, 1, 2, \ldots, m \neq n, \tag{II-8}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \text{where } m,n = 0, 1, 2, \ldots, m \neq n, \tag{II-9}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0, \quad \text{where } m,n = 0, 1, 2, \ldots . \tag{II-10}$$

Since the property of orthogonality has been established, F(x) can be written as a representation in series form with little difficulty arising in the determination of the coefficients. Hence,

$$F(x) = \sum_{m=0}^{\infty} a_m \cos mx + \sum_{n=0}^{\infty} b_n \sin nx.$$

Since all the terms can be obtained when m and n are replaced by a single variable, F(x) can be written as

$$\sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx).$$

However, when $k = 0$; $\cos kx = 1$ and $\sin kx = 0$; hence,

$$F(x) = A_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{II-11}$$

After determination of the coefficients A_0, a_k, and b_k, F(x) can be represented in series form, in this case the Fourier series.

<u>Determination of the coefficients.</u> If it is assumed for purposes of formal calculation that the series can be integrated term by term, integration of (II-11) with the use of the fact that

$$\text{(II-12)} \quad \int_{-\pi}^{\pi} \cos kx \, dx = 0, \qquad \int_{-\pi}^{\pi} \sin kx \, dx = 0, \qquad k \neq 0,$$

$$\int_{-\pi}^{\pi} \cos kx \, dx = 2 \, , \qquad \int_{-\pi}^{\pi} \sin kx \, dx = 0, \qquad k = 0,$$

gives

$$\int_{-\pi}^{\pi} F(x) \, dx = 2\pi A_0,$$

or

$$\text{(II-13)} \qquad A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) \, dx,$$

each integral on the right, with the exception of that of the constant term, reduces to zero. For the determination of a_k when $k \neq 0$, let the identity (II-8) be multiplied through by cos kx, and let the resulting expression for f(x) cos kx be integrated from $-\pi$ to π, still under the assumption that integration term by term is legitimate. Again each integral on the right reduces to zero, by (II-9) and (II-10), except the integral containing $\cos^2 kx$, and it is found that

$$\int_{-\pi}^{\pi} F(x) \cos kx \, dx = a_k \int_{-\pi}^{\pi} \cos^2 kx \, dx = \pi a_k,$$

or

$$\text{(II-14)} \qquad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos kx \, dx.$$

Similarly, multiplication of (II-8) by sin kx, and if the above procedure is followed,

$$\int_{-\pi}^{\pi} F(x) \sin kx \, dx = b_k \int_{-\pi}^{\pi} \sin^2 kx \, dx = \pi b_k,$$

or

$$\text{(II-15)} \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin kx \, dx.$$

Then (II-11) can be written as

$$\text{(II-16)} \qquad F(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x') \, dx'$$

$$+ \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\cos nx \int_{-\pi}^{\pi} F(x') \cos nx' \, dx' \right.$$

$$\left. + \sin nx \int_{-\pi}^{\pi} F(x') \sin nx' \, dx' \right].$$

<u>Expansion of an arbitrary function i_ series</u>. Let the arbitrary function be

$$f(x) = 0, \quad -1 < x < 0; \qquad f(x) = 1, \quad 0 < x < 1.$$

With a substitution of a new variable z in (II-16), where $z = \frac{x}{\pi}$, $z' = \frac{x'}{\pi}$, and substitution of $f(z)$ for $F(\pi z)$, (II-16) becomes

$$f(z) = \frac{1}{2} \int_{-1}^{1} f(z') \, dz' + \sum_{n=1}^{\infty} \left[\cos n\pi z \int_{-1}^{1} f(z') \cos n\pi z' \, dz' \right.$$

$$\left. + \sin n\pi z \int_{-1}^{1} f(z') \sin n\pi z' \, dz' \right].$$

Substitution of the arbitrary function and the change of z to x, which is a different x than that in (II-16), give

$$f(x) = \frac{1}{2}\left[\int_{-1}^{0} (0)\ dx' + \int_{0}^{1} (1)\ dx'\right]$$

$$+ \sum_{n=1}^{\infty}\left\{\cos n\pi x\left[\int_{-1}^{0} (0) \cos n\pi x'\ dx' + \int_{0}^{1} (1) \cos n\pi x'\ dx'\right]\right.$$

$$\left. + \sin n\pi x\left[\int_{-1}^{0} (0) \sin n\pi x'\ dx' + \int_{0}^{1} (1) \sin n\pi x'\ dx'\right]\right\}$$

$$= \frac{1}{2}\Big[x\Big]_{0}^{1} + \sum_{n=1}^{\infty}\left\{\cos n\pi x\left[\frac{1}{n\pi}\sin n\pi x'\right]_{0}^{1} - \sin n\pi x\left[\frac{1}{n\pi}\cos n\pi x'\right]_{0}^{1}\right\}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left[(-1)^{n+1} \sin n\pi x + \sin n\pi x\right],$$

since $\cos n\pi = -1$, when n is odd,

$\cos n\pi = +1$, when n is even.

Then,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left[(-1)^{n+1} + 1\right] \sin n\pi x,$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin \pi x + \frac{2}{3\pi}\sin 3\pi x + \ldots, \qquad -1 < x < 1.$$

A graphical form showing the first few terms of f(x) is represented in Figure 1.

Expansion of a second arbitrary function in series. Let the second arbitrary function be

$$f(x) = -x,\ -1 < x < 0; \qquad f(x) = x,\ 0 < x < 1.$$

$$f(x) = \frac{1}{2}\int_{-1}^{1} f(x')\ dx' + \sum_{n=1}^{\infty}\left[\cos n\pi x \int_{-1}^{1} f(x') \cos n\pi x'\right.$$

$$\left. + \sin n\pi x \int_{-1}^{1} f(x') \sin n\pi x'\ dx'\right]$$

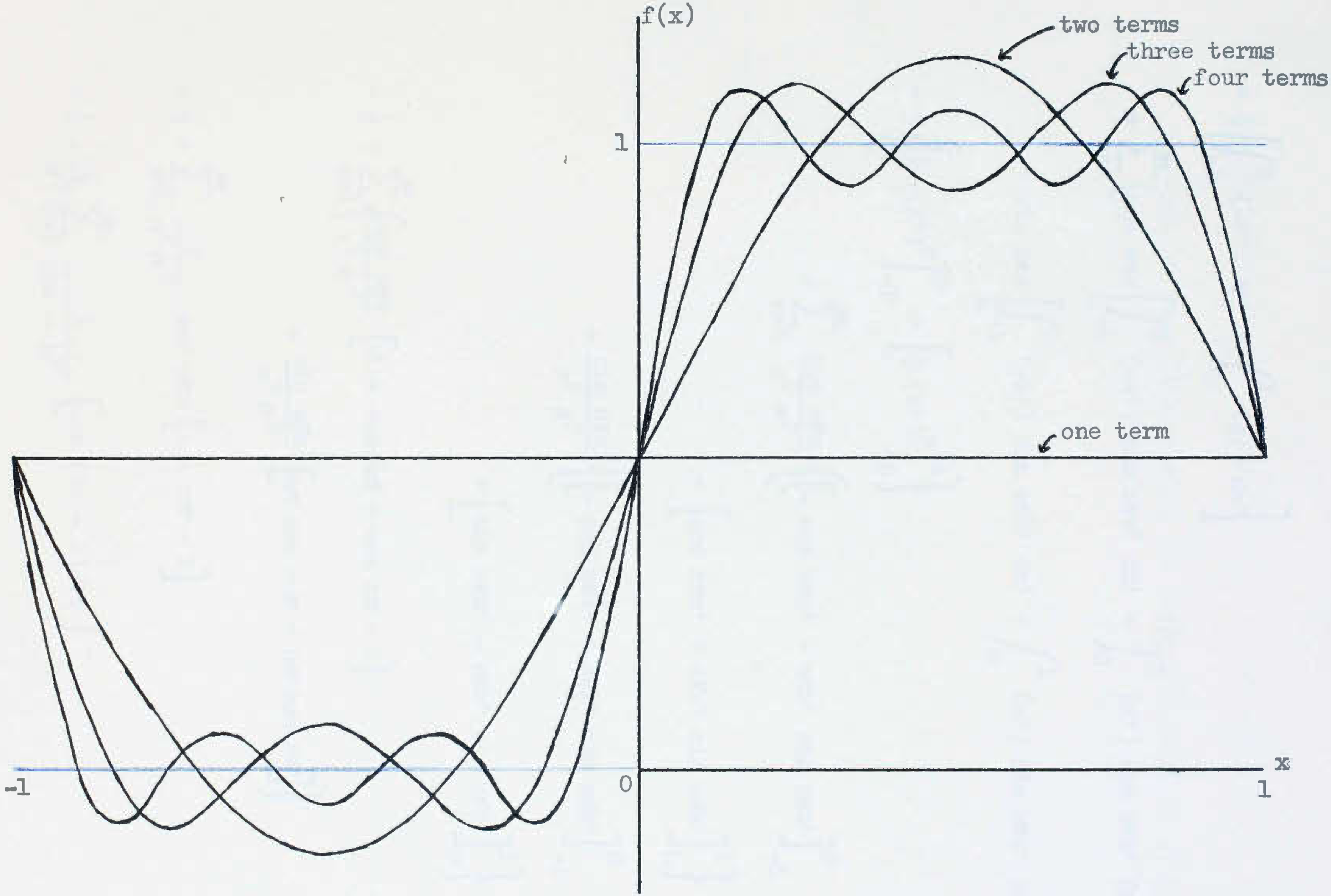

FIGURE 1

A Graphic Representation of the Fourier Series: $f(x) = 0, -1 < x < 0; f(x) = 1, 0 < x < 1.$

$$= \frac{1}{2}\left[\int_{-1}^{0} (-x')\, dx' + \int_{0}^{1} (x')\, dx'\right]$$

$$+ \sum_{n=1}^{\infty}\left\{\cos n\pi x \left[\int_{-1}^{0} (-x') \cos n\pi x'\, dx' + \int_{0}^{1} (x') \cos n\pi x'\, dx'\right]\right.$$

$$\left. + \sin n\pi x \left[\int_{-1}^{0} (-x') \sin n\pi x'\, dx' + \int_{0}^{1} (x') \sin n\pi x'\, dx'\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{2} -(x')^2\right]_{-1}^{0} + \left[\frac{1}{2} (x')^2\right]_{0}^{1}\right\}$$

$$+ \sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2 \pi^2} \left\{\left[- \cos n\pi x' - n\pi x' \sin n\pi x'\right]_{-1}^{0}\right.$$

$$\left. + \left[\cos n\pi x' + n\pi x' \sin n\pi x'\right]_{0}^{1}\right\}$$

$$+ \frac{\sin n\pi x}{n^2 \pi^2} \left\{\left[- \sin n\pi x' + n\pi x' \cos n\pi x'\right]_{-1}^{0}\right.$$

$$\left. + \left[\sin n\pi x' - n\pi x' \cos n\pi x'\right]_{0}^{1}\right\}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left\{\frac{\cos n\pi x}{n^2 \pi^2} \left[-1 + \cos n\pi + \cos n\pi - 1\right]\right.$$

$$\left. + \frac{\sin n\pi x}{n^2 \pi^2} \left[n\pi \cos -n\pi - n\pi \cos n\pi\right]\right\}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \cos n\pi x \left[\cos n\pi - 1\right]$$

$$= \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2} \left[\cos (2n - 1) \pi x\right]$$

$$= \frac{1}{2} - \frac{4}{\pi^2} \cos \pi x - \frac{4}{9\pi^2} \cos 3\pi x - \cdots, \qquad -1 < x < 1.$$

A graphical form illustrating the first few terms of f(x) is shown in Figure 2.

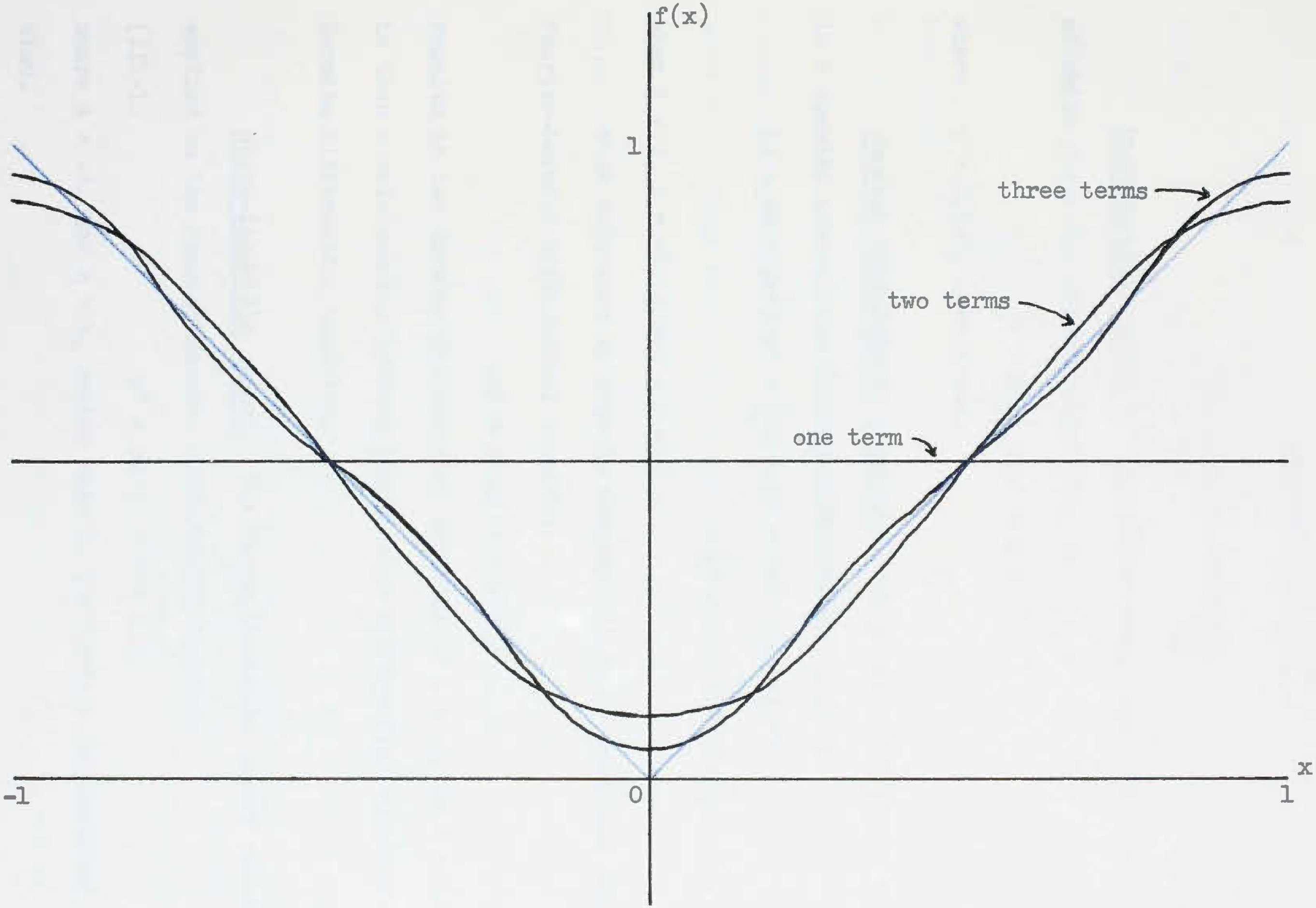

FIGURE 2

A Graphic Representation of the Fourier Series: $f(x) = -x,\ -1 < x < 0;\ f(x) = x,\ 0 < x < 1.$

CHAPTER III

THE HERMITE FUNCTIONS

Differential equation. The differential equation, the solution of which gives the Hermite functions, is

$$y'' - x\,y' + n\,y = 0,$$

where $y = H_n(x)$; $-\infty < x < \infty$.

General differential equation. The above differential equation is a special case of the Pearson differential equation

$$(A + Bx + Cx^2)\,y'' + \Big[(B + D) + (2C + E)\,x\Big]\,y' - \Big[C\,n(n + 1) + E\,n\Big]\,y = 0,$$

when $A = 1$, $E = -1$, $B = C = D = 0$.

With reference to page 5, Chapter II, it is seen that the Fourier-Hermite differential equation

$$y'' + Ax\,y' + B\,y = 0$$

results in the Hermite differential equation if $A = -1$, and $B = n$. There is then a relationship between the Fourier differential equation and the Hermite differential equation.

Sturm-Liouville theory. The Sturm-Liouville theory can be applied to the Fourier-Hermite differential equation

$$\text{(III-1)} \qquad y'' + Ax\,y' + B\,y = 0,$$

where $A = -1$, and $B = n$, rather than to the Hermite differential equation.

It will be noted that

$$f_1(x) = Ax;$$

hence

$$r(x) = e^{\int f_1(x)\,dx} = e^{\int Ax\,dx} = e^{\frac{Ax^2}{2}}.$$

Now (III-1) can be written as

$$\frac{d}{dx}\left[e^{Ax^2/2}\frac{dy}{dx}\right] + B\,e^{Ax^2/2}\,y = 0,$$

from which

(III-2) $$q(x) = 0; \qquad \lambda = B; \quad \text{and } p(x) = e^{Ax^2/2}.$$

With reference to the Sturm-Liouville theory of the Fourier differential equation, where, for the differential equation (II-1) $A = 0$, and $B = n^2$, these values when substituted into (III-2) give

$$q(x) = 0; \qquad \lambda = n^2; \quad \text{and } p(x) = e^0 = 1.$$

These values are noted as the same as that obtained on page 6, Chapter II.

If $A = -1$, and $B = n$ in (III-2), q, p, and λ for the Hermite differential equation are

$$q(x) = 0; \qquad \lambda = n; \qquad \text{and } p(x) = e^{-x^2/2},$$

which are the same as would have been obtained if the Sturm-Liouville theory had been applied directly to the Hermite differential equation.

The weight function is $p(x) = e^{-x^2/2}$, and for the Hermite functions to be orthogonal over the interval $(-\infty, \infty)$, the functions must satisfy the following equation:

(III-3) $$\int_{-\infty}^{\infty} e^{-x^2/2}\,H_m(x)\,H_n(x)\,dx = 0, \qquad m \neq n.$$

The $H(x)$'s can be found if $H_0(x) = a_0$, $H_1(x) = a_1 + b_1x$, $H_2(x) = a_2 + b_2x + c_2x^2$, ..., and the various constants evaluated. If $a_0 = 1$, it is found that

(III-4)
$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= x \\
H_2(x) &= x^2 - 1 \\
H_3(x) &= x^3 - 3x \\
H_4(x) &= x^4 - 6x^2 + 3 \\
H_5(x) &= x^5 - 10x^3 + 15x \\
H_6(x) &= x^6 - 15x^4 + 45x^2 - 15 \\
H_7(x) &= x^7 - 21x^5 + 105x^3 - 105\,x \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

There is, however, another approach to the problem of finding the $H(x)$'s.[1]

Let

$$\phi(x) = e^{-x^2/2}.$$

By straightforward differentiation,

$$\phi'(x) = -x\, e^{-x^2/2}, \qquad \phi''(x) = (x^2 - 1)\, e^{-x^2/2},$$

$$\phi'''(x) = (-x^3 + 3x)\, e^{-x^2/2}, \ \ldots .$$

It will be noted that the derivative of any order is the product of $e^{-x^2/2}$ by a polynomial in x.

(III-5)
$$H_n(x) = (-1)^n\, e^{x^2/2}\, \frac{d^n}{dx^n}\, \phi(x),$$

---

[1]Jackson, op. cit., p. 176.

and substitution of various n's will give the same values for $H_n(x)$, $n = 0, 1, 2, \ldots$, as in (III-4).

If the various H(x)'s are written as a series representation of y or f(x):

$$f(x) = a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + \ldots + a_n H_n(x) + \ldots = \sum_{n=0}^{\infty} a_n H_n(x), \qquad -\infty < x < \infty. \tag{III-6}$$

Determination of the coefficients. For the determination of a_n let (III-6) be multiplied through by $H_n(x)$ and $e^{-x^2/2}$, and let the resulting expression for $e^{-x^2/2} f(x) H_n(x)$ be integrated from $-\infty$ to ∞, under the assumption that integration term by term is legitimate. Each integral of the right reduces to zero, by (III-3), except the integral containing $H_n^2(x)$. It is then found that

$$\int_{-\infty}^{\infty} e^{-x^2/2} H_n(x) f(x)\, dx = a_n \int_{-\infty}^{\infty} e^{-x^2/2} H_n^2(x)\, dx,$$

where $\int_{-\infty}^{\infty} e^{-x^2/2} H_n^2(x)\, dx$ is found[2] to equal

$$n!\,(2\pi)^{\frac{1}{2}}.$$

Hence

$$a_n = \frac{1}{n!\,(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-x^2/2} H_n(x) f(x)\, dx. \tag{III-7}$$

[2] Ibid., p. 178.

<u>Expansion of an arbitrary function in series.</u> Let the arbitrary function be

$$f(x) = 0, \ -\infty < x < 0; \ f(x) = 1, \ 0 < x < \infty.$$

Substitution of (III-7) into (III-6) discloses that

$$f(x) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n! \ (2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-x'^2/2} H_n(x') \ f(x') \ dx'.$$

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n! \ (2\pi)^{\frac{1}{2}}} \left[\int_{-\infty}^{0} e^{-x'^2/2} H_n(x') \ (0) \ dx' + \int_{0}^{\infty} e^{-x'^2/2} H_n(x') \ dx' \right].$$

By use of (III-5)

$$\int_0^{\infty} e^{-x'^2/2} H_n(x') \ dx' = \int_0^{\infty} (-1)^n \frac{d^n \phi(x)}{dx^n} dx$$

$$= \left[(-1)^n \frac{d^{n-1} \phi(x)}{dx^n} \right]_0^{\infty}.$$

However, $H_{n-1}(x) = (-1)^{n-1} e^{x^2/2} \dfrac{d^{n-1} \phi(x)}{dx^{n-1}}$,

so $$\left[(-1)^n \frac{d^{n-1} \phi(x)}{dx^{n-1}} \right]_0^{\infty} = \left[(-1) \ e^{-x^2/2} H_{n-1}(x) \right]_0^{\infty} = \left[- 0 + H_{n-1}(0) \right]$$

$$= H_{n-1}(0).$$

However by (III-4), it is noted that when $n = 2, 4, \ldots, H_{n-1}(0)$ will drop out. There is one more consideration to be taken, and that occurs when $n = 0$, since $H_{-1}(0)$ is undefined by (III-4). Therefore, it is necessary to substitute $n = 0$ in equation (III-7), whence

$$\int_0^{\infty} e^{-x'^2/2} H_0(x') \cdot (1) \ dx' = \int_0^{\infty} e^{-x'^2/2} dx' = \frac{\sqrt{2\pi}}{2}.$$

Now f(x) can be written in series form as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{H_{n-1}(0) \quad H_n(x)}{n! \; (2\pi)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{2} H_0(x) + \frac{1}{\sqrt{2\pi}} H_1(x) - \frac{1}{3! \sqrt{2\pi}} H_3(x) + \ldots$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} (x) - \frac{1}{3! \sqrt{2\pi}} (x^3 - 3x) + \frac{3}{5! \sqrt{2\pi}} (x^5 - 10x^3 - 15x) - \ldots .$$

A graphic illustration of the first few terms of the above series is shown in Figure 3, with the interval (-5,5). This small interval was taken in order to reduce the size of the graph.

<u>Expansion of a second arbitrary function in series.</u> Let the second function be

$$f(x) = -x, \; -\infty < x < 0; \; f(x) = x, \; 0 < x < \infty.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n! \; (2\pi)^{\frac{1}{2}}} \left[\int_{-\infty}^{0} e^{-x'^2/2} \; H_n(x') \; (-x') \; dx' \right.$$

$$\left. + \int_0^{\infty} e^{-x'^2/2} \; H_n(x') \; (x') \; dx' \right]$$

$$= \sum_{n=0}^{\infty} \frac{H_n(x)}{n! \; (2\pi)^{\frac{1}{2}}} \left[\int_0^{-\infty} x' \; e^{-x'^2/2} \; H_n(x') \; dx' + \int_0^{\infty} x' \; e^{-x'^2/2} \; H_n(x') \; dx' \right].$$

By the substitution of $x' = -z$ in the first integral,

$$f(x) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n! \; (2\pi)^{\frac{1}{2}}} \left[-\int_0^{\infty} z \; e^{-z^2/2} \; H_n(-z) \; (-dz) + \int_0^{\infty} x' \; e^{-x'^2/2} \; H_n(x') \; dx' \right],$$

or, since z is a dummy variable,

$$f(x) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n! \; (2\pi)^{\frac{1}{2}}} \left[\int_0^{\infty} x' \; e^{-x'^2/2} \; H_n(-x') \; dx' + \int_0^{\infty} x' \; e^{-x'^2/2} \; H_n(x') \; dx' \right].$$

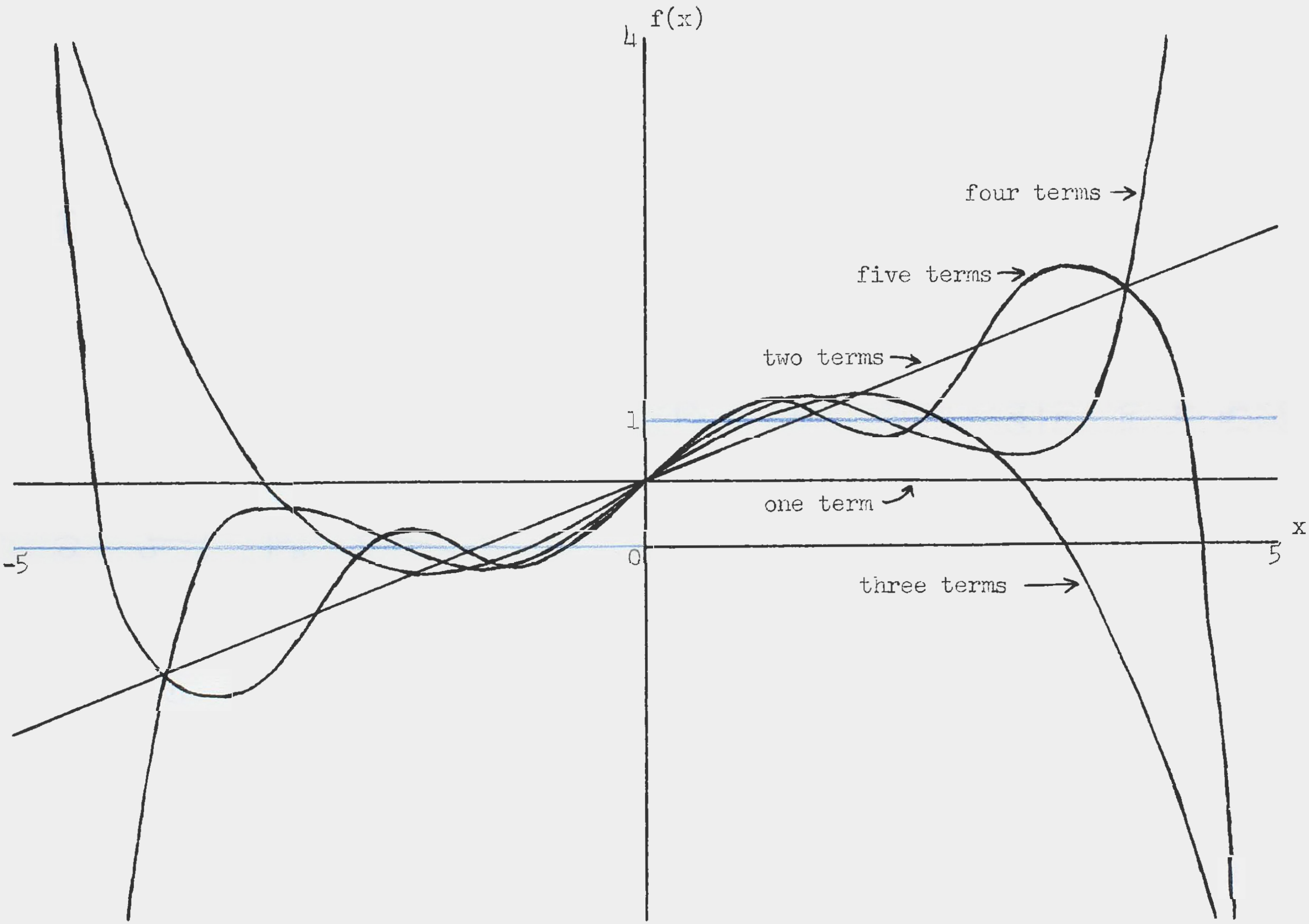

FIGURE 3

A Graphic Representation of the Hermite Series: $f(x) = 0,\ -5 < x < 0;\ f(x) = 1,\ 0 < x < 5.$

However, when n is odd, $H_n(x)$ is an odd function; therefore $H_n(-x) = - H_n(x)$. When n is even, $H_n(x)$ is an even function; therefore $H_n(-x) = H_n(x)$. Hence, when n is odd $a_n = 0$, since the integrals cancel, and when n is even, since the integrals add,

$$\text{(III-8)} \qquad f(x) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!\ (2\pi)^{\frac{1}{2}}}\ 2\int_0^{\infty} x'\ e^{-x'^2/2}\ H_n(x')\ dx', \quad (n,\ \text{even}).$$

By use of (III-5)

$$\int_0^{\infty} x'\ e^{-x'^2/2}\ H_n(x')\ dx' = \int_0^{\infty} (-1)^n\ x\ \frac{d^n\ \phi(x)}{dx^n}\ dx.$$

The integral on the right is integrated by parts, where

$$du = dx, \qquad u = x$$

$$dv = \frac{d^n\ \phi(x)}{dx^n}\ dx, \qquad v = \frac{d^{n-1}\ \phi(x)}{dx^{n-1}}\ .$$

Then

$$\int_0^{\infty} x'\ e^{-x'^2/2}\ H_n(x')\ dx'$$

$$= \left[(-1)^n\ x\ \frac{d^{n-1}\ \phi(x)}{dx^{n-1}}\right]_0^{\infty} - \left[(-1)^n \int_0^{\infty} \frac{d^{n-1}\ \phi(x)}{dx^{n-1}}\ dx\right]$$

$$= \left[(-1)^n\ x\ \frac{d^{n-1}\ \phi(x)}{dx^{n-1}}\right]_0^{\infty} - \left[(-1)^n\ \frac{d^{n-2}\ \phi(x)}{dx^{n-2}}\right]_0^{\infty}$$

$$= \left[(-1)^n(-1)^{n-1}\ xe^{-x^2/2}\ H_{n-1}(x)\right]_0^{\infty} - \left[(-1)^n(-1)^{n-2}\ e^{-x^2/2}\ H_{n-2}(x)\right]_0^{\infty}$$

$$= \left[0 + (-1)^{2n-2}\ H_{n-2}(0)\right] = H_{n-2}(0), \quad (n,\ \text{even}),$$

since $xe^{-x^2/2}\ H_n(x)$ can be likened to $x^m\ e^{-x^2/2}$; and

$$\lim_{x\to\infty} \frac{x^m}{e^{x^2/2}} = \lim_{x\to\infty} \frac{mx^{m-1}}{e^{x^2/2}} = \lim_{x\to\infty} \frac{mx^{m-2}}{e^{x^2/2}} = \lim_{x\to\infty} \frac{(m)(m-2)(m-4)\ \dots\ x}{xe^{x^2/2}} = 0.$$

Now

$$H_n(0) = (-1)^{n/2}\ 1\cdot3\cdot5\ \dots\ (n-1);$$

so

$$H_{n-2}(0) = (-1)^{(n-2)/2}\ \frac{1\cdot3\cdot5\ \dots\ (n-3)\ 2\cdot4\cdot6\ \dots\ (n-2)}{2\cdot4\cdot6\ \dots\ (n-2)}$$

$$= \frac{(-1)^{(n-2)/2}\ (n-2)!}{2\cdot4\cdot6\ \dots\ (n-2)} = \frac{(-1)^{(n-2)/2}\ (n-2)!\ (n-1)\ n}{2^{(n-2)/2}\ \left(\frac{n-2}{2}\right)!\ (n-1)\ n}$$

$$= \frac{(-1)^{(n-2)/2}\ n!}{2^{(n-2)/2}\ n(n-1)\left(\frac{n-2}{2}\right)!}, \qquad (n,\ \text{even}).$$

It will be noted, however, that when $n = 0$, $H_{n-2}(0)$ is undefined, so by use of (III-8), $H_{-2}(0)$ can be determined as

$$\int_0^\infty xe^{-x^2/2}\ H_0(x)\ dx = \int_0^\infty xe^{-x^2/2}\ dx = \frac{1}{2\ (\frac{1}{2})} = 1.$$

Hence,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{(n-2)/2}}{2^{(n-4)/2}}\ \frac{H_n(x)}{(2\pi)^{\frac{1}{2}}\ n(n-1)\left(\frac{n-2}{2}\right)!}, \qquad (n,\ \text{even})$$

$$= \frac{2}{\sqrt{2\pi}}\ H_0(x) + \frac{1}{\sqrt{2\pi}}\ H_2(x) - \frac{1}{12\sqrt{2\pi}}\ H_4(x) + \dots .$$

Figure 4 illustrates a graphic representation of the above series with interval (-5,5).

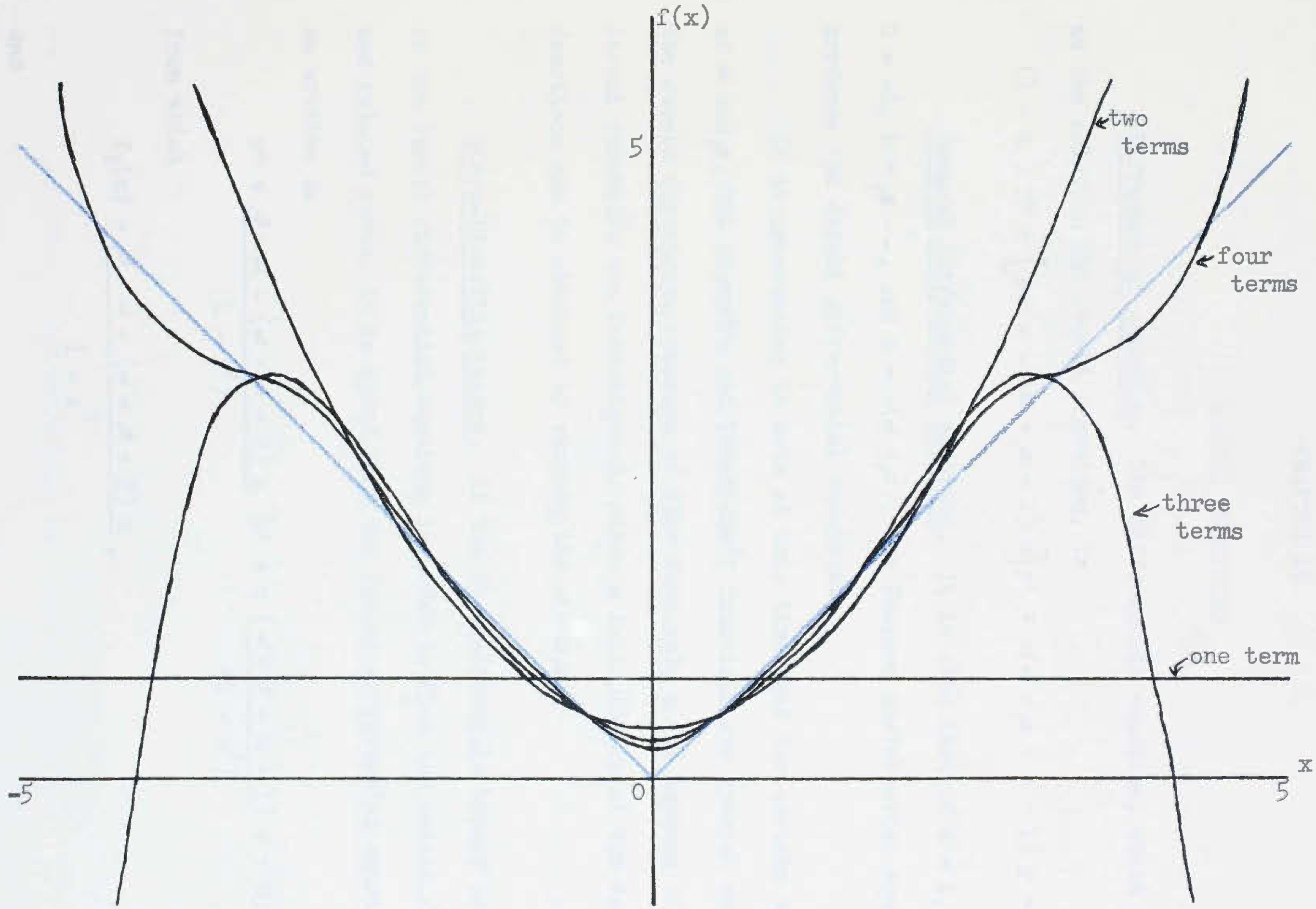

FIGURE 4

A Graphic Representation of the Hermite Series: $f(x) = -x$, $-5 < x < 0$; $f(x) = x$, $0 < x < 5$.

CHAPTER IV

JACOBI FUNCTIONS

Differential equation. The differential equation, which gives as the solution the Jacobi functions, is

$$(1 - x\ )\, y'' + \left[\beta - \alpha - (\alpha + \beta + 2)\, x\right] y' + n(\alpha + \beta + n + 1)\, y = 0.$$

General differential equation. It is seen that if $A = 1$, $B = 0$, $C = -1$, $D = \beta - \alpha$, and $E = -(\alpha + \beta)$, the Pearson differential equation produces the Jacobi differential equation.

It is interesting to note at this time that for certain values of α and β, the Legendre and Tchebicheff functions are special cases of the Jacobi functions. Because of this fact only a few aspects of the Jacobi functions are investigated, since a full picture of the Jacobi functions can be obtained by varying the α and β.

Sturm-Liouville theory. If the Sturm-Liouville theory is applied to the Jacobi differential equation in order to find the weight function and related parts, it is noted that the Jacobi differential equation can be written as

$$y'' + \frac{\beta - \alpha - (\alpha + \beta + 2)\, x}{(1 - x^2)}\, y' + n\, \frac{(\alpha + \beta + n + 1)}{(1 - x^2)}\, y = 0,$$

from which

$$f_1(x) = \frac{\beta - \alpha - (\alpha + \beta + 2)\, x}{1 - x^2},$$

and

$$r(x) = e^{\int \frac{\beta - \alpha - (\alpha + \beta + 2)\, x}{1 - x^2}\, dx}$$

$$= e^{\int \frac{\beta\, dx}{1-x^2} - \int \frac{\alpha\, dx}{1-x^2} - \int \frac{\alpha x\, dx}{1-x^2} - \int \frac{\beta x\, dx}{1-x^2} - \int \frac{2x\, dx}{1-x^2}}$$

$$= e^{\left[\frac{\beta}{2} \ln \left|\frac{1+x}{1-x}\right| - \frac{\alpha}{2} \ln \left|\frac{1+x}{1-x}\right| + \frac{\alpha}{2} \ln \left|1-x^2\right| + \frac{\beta}{2} \ln \left|1-x^2\right| + \ln \left|1-x^2\right|\right]}$$

$$= e^{\ln \left[\left[\left|\frac{1+x}{1-x}\right| \left|(1+x)(1-x)\right|\right]^{\frac{\beta}{2}} \left[\left|\frac{1-x}{1+x}\right| \left|(1+x)(1-x)\right|\right]^{\frac{\alpha}{2}} \left|1-x^2\right|\right]}$$

$$= \left|1 + x\right|^{\beta} \left|1 - x\right|^{\alpha} \left|1 - x^2\right|.$$

Since $-1 < x < 1$

$$r(x) = (1 + x)^{\beta}(1 - x)^{\alpha}(1 - x^2).$$

The Jacobi differential equation can now be written as

$$\frac{d}{dx}\left[(1+x)^{\beta} (1-x)^{\alpha} (1-x^2) \frac{dy}{dx}\right] + n(\alpha + \beta + n + 1)(1+x)^{\beta} (1-x)^{\alpha} y = 0,$$

$$-1 < x < 1,$$

from which

$$q(x) = 0; \qquad \lambda = n(\alpha + \beta + n + 1); \qquad p(x) = (1+x)^{\beta} (1-x)^{\alpha}.$$

The weight function $p(x)$ is $(1+x)^{\beta} (1-x)^{\alpha}$; so if the Jacobi functions are $J_n(x)$, $n = 0, 1, 2, \ldots$, they will be orthogonal as

$$\int_{-1}^{1} p(x)\, J_m(x)\, J_n(x)\, dx = 0, \quad m,n = 0, 1, 2, \ldots , m \neq n,$$

or

$$\text{(IV-1)} \qquad \int_{-1}^{1} (1+x)^{\beta} (1-x)^{\alpha} J_m(x)\, J_n(x)\, dx = 0,$$

where the interval of orthogonality is $(-1,1)$.

The applicability of this aspect of the Jacobi function to the Legendre and Tchebicheff functions will be shown in the chapters that follow.

Determination of the coefficients. Since the various $J_n(x)$'s are determined to be orthogonal, it is with little difficulty that a given function of x can be represented in series form that utilizes these orthogonal functions. This function of x can be written as

$$\text{(IV-2)} \quad f(x) = a_0J_0(x) + a_1J_1(x) + a_2J_2(x) + \dots + a_nJ_n(x) + \dots$$

$$= \sum_{n=0}^{\infty} a_nJ_n(x), \qquad -1 < x < 1.$$

Let (IV-2) be multiplied through by Jn(x) and the weight function $p(x) = (1+x)^{\beta} (1-x)^{\alpha}$, and let the resulting expression for

$$(1+x)^{\beta} (1-x)^{\alpha} f(x) J_n(x)$$

be integrated from -1 to 1, integration term by term is assumed to be legitimate. Each integral on the right reduces to zero, by (IV-1), except the integral containing $J_n^2(x)$. It is then found that

$$\int_{-1}^{1} (1+x)^{\beta} (1-x)^{\alpha} J_n(x) f(x)\, dx = a_n \int_{-1}^{1} (1+x)^{\beta} (1-x)^{\alpha} J_n^2(x)\, dx,$$

or

$$\text{(IV-3)} \quad a_n = \frac{1}{\int_{-1}^{1} (1+x)^{\beta}(1-x)^{\alpha} J_n^2(x)\, dx} \int_{-1}^{1} (1+x)^{\beta}(1-x)^{\alpha} J_n(x) f(x)\, dx.$$

With a substitution for α and β, a_n can then be determined. This will be shown in the succeeding chapters.

CHAPTER V

THE LEGENDRE FUNCTIONS

Differential equation. The Legendre functions are solutions of the Jacobi differential equation where $\alpha = \beta = 0$, or

$$(1 - x^2)\, y'' - 2x\, y' + n(n + 1)\, y = 0. \tag{V-1}$$

General differential equation. With reference to page 26, Chapter IV, and if $\alpha = \beta = 0$, then for the Pearson differential equation to result in the Legendre differential equation A = 1, B = 0, C = -1, D = 0, and E = 0.

Sturm-Liouville theory. Since $\alpha = \beta = 0$,

then

$$f_1(x) = \frac{\beta - \alpha - (\alpha + \beta + 2)\, x}{1 - x^2} = \frac{-2x}{1 - x^2},$$

$$r(x) = (1 + x)^{\beta} (1 - x)^{\alpha} (1 - x^2) = (1 - x^2),$$

$$q(x) = 0, \quad \lambda = n(\alpha + \beta + n + 1) = n(n + 1)$$

$$p(x) = (1 + x)^{\beta} (1 - x)^{\alpha} = 1, \text{ where } -1 < x < 1.$$

Replacement of the $J_n(x)$ of the Jacobi function by $P_n(x)$ for the Legendre function in (IV-1) gives

$$\int_{-1}^{1} (1)\, P_m(x)\, P_n(x)\, dx = 0; \quad m,n = 0, 1, 2, \ldots, m \neq n. \tag{V-2}$$

In order to determine the P(x)'s[1], assume that

[1]Ivan S. and Elizabeth S. Sokolnikoff, Higher Mathematics for Engineers and Physicists, New York, 1941, pp. 342-343.

(V-3) $\quad y = a_0x^m + a_1x^{m+1} + \dots + a_kx^{m+k} + \dots$

is a solution of (V-1). Then

$$y'' = m(m - 1)\ a_0x^{m-2} + (m + 1)m\ a_1x^{m-1} + (m + 2)(m + 1)\ a_2x^m +$$
$$\dots + (m + k)(m + k - 1)\ a_kx^{m+k-2} + \dots ,$$

$$-x^2y'' = -m(m - 1)\ a_0x^m - \dots - (m + k - 2)(m + k - 3)\ a_{k-2}x^{m+k-2}$$
$$- \dots ,$$

$$-2xy' = -2m\ a_0x^m - \dots - 2(m + k - 2)\ a_{k-2}x^{m+k-2} - \dots ,$$

$$n(n + 1)\ y = n(n + 1)\ a_0x^m + \dots + n(n + 1)\ a_{k-2}x^{m+k-2} + \dots .$$

Addition of these expressions and equation to zero the coefficients of x^{m-2}, x^{m-1}, ... , x^{m+k-2} give the system of equations

$$m(m - 1)\ a_0 = 0,$$

$$m(m + 1)\ a_1 = 0,$$

$$\dots\dots\dots\dots\dots ,$$

$$(m + k)(m + k - 1)\ a_k + (n - m - k + 2)(n + m + k - 1)\ a_{k-2} = 0.$$

In order to satisfy the first of these equations, m can be taken as either 0 or 1. If m = 1, the second equation requires that $a_1 = 0$. For m = 0 the coefficients of x^{m+k-2} gives the recursion formula

$$a_k = -\frac{(n - k + 2)(n + k - 1)}{k(k - 1)}\ a_{k-2},$$

from which the coefficients a_2, a_3, a_4, ... can be determined. If m = 0, the second of the equations of the system allows a_1 to be arbitrary.

If the values of the coefficients in terms of a_0 and a_1 are substituted in (V-3), the following solution is obtained:

$$\text{(V-4)} \qquad y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right]$$
$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\right].$$

By means of the ratio test[2] it can be shown that, for non-integral values of n, the interval of convergence of the series in (V-4) is $(-1,1)$. Moreover, since the first series in (V-4) represents an even function and the second series represents an odd function, the two solutions are linearly independent. Multiplication of each of the two series by an arbitrary constant gives the general solution of (V-1), which is valid if $-1 < x < 1$. When n is an even integer, the first series in (V-4) terminates and reduces to a polynomial, whereas, when n is an odd integer, the second series becomes a polynomial. If a_0 and a_1 are so adjusted as to give these polynomials the value unity when $x = 1$, then the following set of polynomials is obtained:

$$\text{(V-5)} \qquad \begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2}, \\ P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x, \\ P_4(x) &= \frac{7\cdot5}{4\cdot2}x - 2\,\frac{5\cdot3}{4\cdot2}x + \frac{3\cdot1}{4\cdot2}, \end{aligned}$$

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[2] Ibid., p. 31.

$$P_5(x) = \frac{9\cdot7}{4\cdot2}\, x^5 - 2\, \frac{7\cdot5}{4\cdot2}\, x^3 + \frac{5\cdot3}{4\cdot2}\, x,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where the subscripts on P indicate the value n.

Determination of the coefficients. Since the Legendre functions are special cases of the Jacobi functions when $\alpha = \beta = 0$, then by (IV-3), a_n of the series

(V-6) $$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \qquad -1 < x < 1$$

equals

$$\frac{1}{\int_{-1}^{1} P_n^2(x)\, dx} \int_{-1}^{1} P_n(x)\, f(x)\, dx,$$

where

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$ [3]

Thus

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x)\, f(x)\, dx.$$

Expansion of an arbitrary function in series. Let the arbitrary function be

$$f(x) = 0, \ -1 < x < 0; \ f(x) = 1, \ 0 < x < 1.$$

By (V-6),

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x)\, f(x)\, dx$$

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[3] Jackson, op. cit., p. 52.

$$= \frac{2n+1}{2}\left[\int_{-1}^{0} P_n(x)\,(0)\,dx + \int_{0}^{1} P_n(x)\,(1)\,dx\right]$$

$$= \frac{2n+1}{2}\int_{0}^{1} P_n(x)\,dx.$$

Now,

(V-7) $$P_n(x) = \frac{P'_{n+1}(x) - P'_{n-1}(x)}{2n+1}.$$ [4]

Then

$$a_n = \tfrac{1}{2}\int_{0}^{1}\left[P'_{n+1}(x) - P'_{n-1}(x)\right]dx$$

$$= \tfrac{1}{2}\left[P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(0) + P_{n-1}(0)\right].$$

However,

$$P_{n+1}(1) = P_{n-1}(1),$$

from the development of (V-5). Hence,

$$a_n = -\tfrac{1}{2}\left[P_{n+1}(0) - P_{n-1}(0)\right].$$

By (V-5) $P_n(0) = 0$, for n = odd integers, then $a_n = 0$, for n = even integers. If n is replaced by $2n + 1$; $n + 1$ by $2n + 2$; $n - 1$ by $2n$, then

$$a_{2n+1} = -\tfrac{1}{2}\left[P_{2n+2}(0) - P_{2n}(0)\right].$$

However,

$$P_{2n}(0) = (-1)^n \frac{1\cdot3\cdot5\ \dots\ (2n-1)}{2\cdot4\cdot6\ \dots\ (2n)},$$ [5]

and

$$P_{2n+2}(0) = (-1)^{n+1} \frac{1\cdot3\cdot5\ \dots\ (2n+1)}{2\cdot4\cdot6\ \dots\ (2n+2)},$$

whence

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[4] Ibid., p. 49.

[5] Churchill, op. cit., p. 180.

$$a_{2n+1} = -\tfrac{1}{2}\left[(-1)^{n+1}\frac{1\cdot3\cdot5\ \dots\ (2n+1)}{2\cdot4\cdot6\ \dots\ (2n+2)} - (-1)^n\frac{1\cdot3\cdot5\ \dots\ (2n-1)}{2\cdot4\cdot6\ \dots\ (2n)}\right]$$

$$= -\tfrac{1}{2}\left[(-1)^n\frac{1\cdot3\cdot5\ \dots\ (2n-1)}{2\cdot4\cdot6\ \dots\ (2n)}\left(-\ \frac{2n+1}{2n+2} - 1\right)\right]$$

$$= \frac{(-1)^n}{2}\,\frac{1\cdot3\cdot5\ \dots\ (2n-1)}{2\cdot4\cdot6\ \dots\ (2n)}\cdot\frac{2n+1+2n+2}{2n+2}$$

$$= \frac{(-1)^n}{2}\,\frac{1\cdot3\cdot5\ \dots\ (2n-1)(4n+3)}{2\cdot4\cdot6\ \dots\ (2n+2)}\ .$$

Then

$$a_{2n-1} = \frac{(-1)^{n-1}}{2}\,\frac{1\cdot3\cdot5\ \dots\ (2n-3)(4n-1)}{2\cdot4\cdot6\ \dots\ (2n)}\ ,$$

or

(V-8) $$a_{2n+1} = (-1)\ \frac{(2n-1)(4n+3)}{(2n+2)(4n-1)}\ a_{2n-1}\ .$$

Now

$$a_0 = \frac{2(0)+1}{2}\int_0^1 (1)\ P_0(x)\ dx = \tfrac{1}{2}\int_0^1 dx = \tfrac{1}{2}\Big[x\Big]_0^1 = \tfrac{1}{2}, \text{ and}$$

$$a_1 = \frac{2+1}{2}\int_0^1 (1)\ P_1(x)\ dx = \frac{3}{2}\int_0^1 x\ dx = \frac{3}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{3}{4}\ .$$

Then the use of the recursion formula (V-8) and the fact that $a_n = 0$, when n is even, gives

$$a_2 = 0,$$

$$a_3 = -\ \frac{1\cdot7}{4\cdot3}\,\frac{3}{4} = -\ \frac{7}{16}\ ,$$

$$a_4 = 0,$$

$$a_5 = -\ \frac{3\cdot11}{6\cdot7}\cdot -\ \frac{7}{16} = \frac{11}{32}\ ,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Now

$$f(x) = \tfrac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + \sum_{n=1}^{\infty} (-1)^n \frac{4n+3}{4n+4} \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)} P_{2n+1}(x)$$

$$= \tfrac{1}{2} + \frac{3}{4} x - \frac{7}{32} (5x^3 - 3x) + \frac{11}{256} (63x^5 - 70x^3 + 15x) - \ldots ,$$

$$-1 < x < 1.$$

Figure 5 shows a graphic representation of the first few terms of the above series.

Expansion of a second arbitrary function in series. For a second arbitrary function let

$$f(x) = -x, \ -1 < x < 0; \ f(x) = x, \ 0 < x < 1.$$

The coefficient

$$a_n = \frac{2n+1}{2}\left[\int_{-1}^{0} (-x) P_n(x)\, dx + \int_{0}^{1} (x) P_n(x)\, dx\right]$$

$$= \frac{2n+1}{2}\left[\int_{0}^{-1} (x) P_n(x)\, dx + \int_{0}^{1} (x) P_n(x)\, dx\right].$$

If $x = -z$ in the first integral,

$$a_n = \frac{2n+1}{2}\left[\int_{0}^{1} (z) P_n(-z)\, dz + \int_{0}^{1} (x) P_n(x)\, dx\right].$$

However, when n is odd, $P_n(x)$ is an odd function; therefore, $P_n(-x) = - P_n(x)$. When n is even, $P_n(x)$ is an even function; therefore $P_n(-x) = P_n(x)$. Hence, when n is odd, $a_n = 0$, since the integrals cancel, and when n is even the integrals add; then

$$\text{(V-9)} \qquad a_n = (2n + 1)\int_{0}^{1} x\, P_n(x)\, dx, \qquad \text{(n, even)}.$$

Then by (V-7)

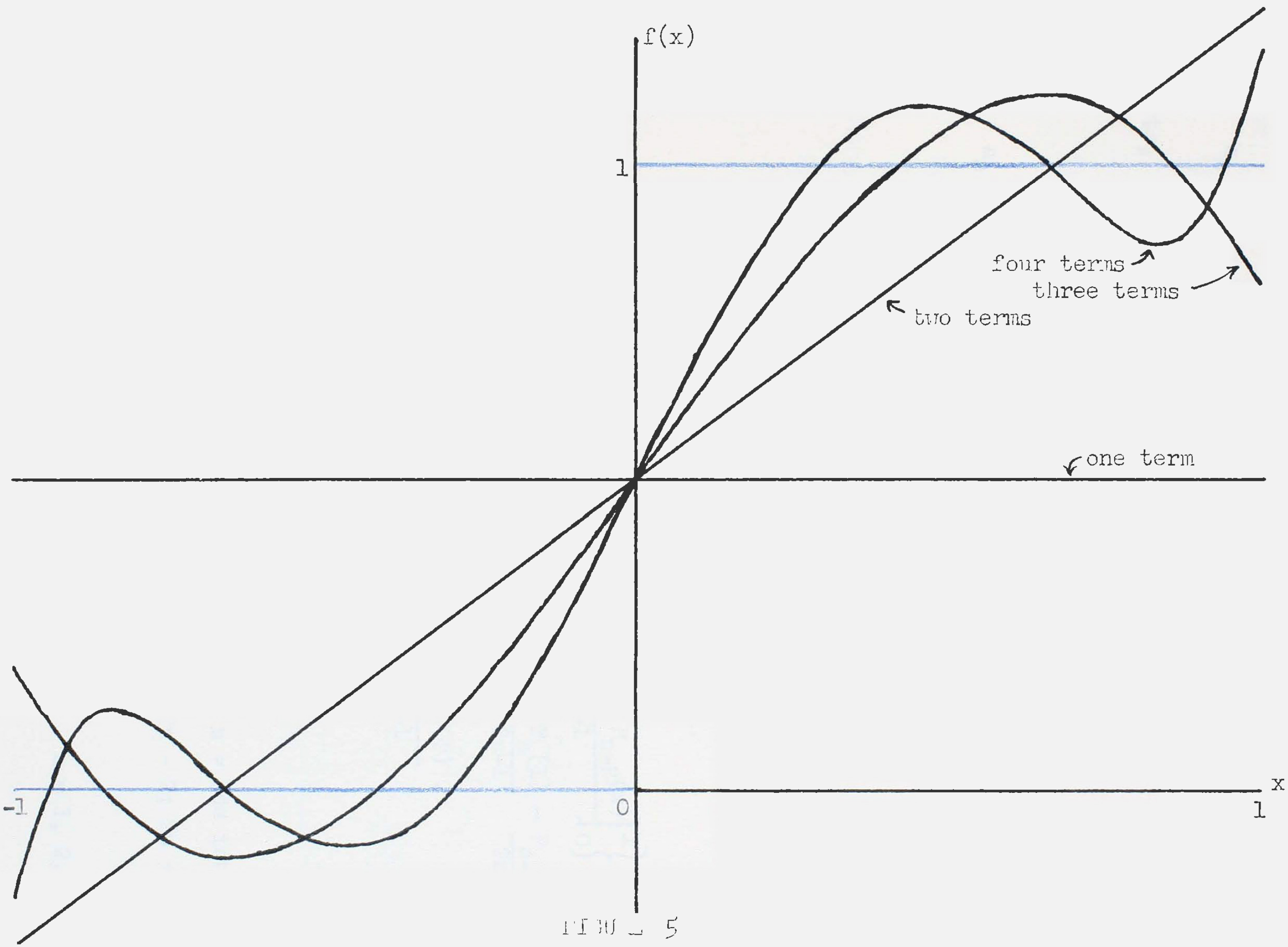

FIGURE 5

A Graphic Representation of the Legendre Series: $f(x) = 0, -1 < x < 0$; $f(x) = 1, 0 < x < 1$.

$$a_n = \int_0^1 x\left[P'_{n+1}(x) - P'_{n-1}(x)\right] dx, \qquad (n,\ \text{even}).$$

By parts,

$$u = x, \qquad du = dx,$$

$$dv = P'_{n+1}(x)\,dx, \qquad v = P_{n+1}(x),$$

$$a_n = \left\{\left[x\,P_{n+1}(x)\right]_0^1 - \int_0^1 P_{n+1}(x)\,dx - \left[x\,P_{n-1}(x)\right]_0^1 + \int_0^1 P_{n-1}(x)\,dx\right\}$$

$$= \left\{\left[x\,P_{n+1}(x)\right]_0^1 - \int_0^1 \frac{P'_{n+2}(x) - P'_n(x)}{2n+3}\,dx\right.$$

$$\left. - \left[x\,P_{n-1}(x)\right]_0^1 + \int_0^1 \frac{P'_n(x) - P'_{n-2}(x)}{2n-1}\,dx\right\}$$

$$= \left\{\left[x\,P_{n+1}(x) - \frac{P_{n+2}(x) - P_n(x)}{2n+3}\right.\right.$$

$$\left.\left. - x\,P_{n-1}(x) + \frac{P_n(x) - P_{n-2}(x)}{2n-1}\right]_0^1\right\}$$

$$= P_{n+1}(1) - \frac{P_{n+2}(1)}{2n+3} + \frac{P_n(1)}{2n+3} - P_{n-1}(1) + \frac{P_n(1)}{2n-1} - \frac{P_{n-2}(1)}{2n-1}$$

$$+ \frac{P_{n+2}(0)}{2n+3} - \frac{P_n(0)}{2n+3} - \frac{P_n(0)}{2n-1} + \frac{P_{n-2}(0)}{2n-1}\,.$$

Since $P_{n'}(1) = 1$,

$$a_n = \frac{P_{n+2}(0)}{2n+3} - \frac{P_n(0)}{2n+3} - \frac{P_n(0)}{2n-1} + \frac{P_{n-2}(0)}{2n-1}\,.$$

Since $P_n(0) = 0$, for n = odd integers, $a_n = 0$, for n = odd integers.

If n is replaced by 2n; n + 2 by 2n + 2; n - 2 by 2n - 2; 2n + 3 by 4n + 3; 2n - 1 by 4n - 1, n = 1, 2, 3, ... , then

$$a_{2n} = \frac{P_{2n+2}(0)}{4n+3} - \frac{P_{2n}(0)}{4n+3} - \frac{P_{2n}(0)}{4n-1} + \frac{P_{2n-2}(0)}{4n-1}\,, \qquad n = 1, 2, 3, \ldots\,.$$

However,

$$P_{2n}(0) = (-1)^n \frac{1\cdot3\cdot5 \ldots (2n-1)}{2\cdot4\cdot6 \ldots (2n)} ;$$

then

$$P_{2n+2}(0) = (-1)^{n+1} \frac{1\cdot3\cdot5 \ldots (2n+1)}{2\cdot4\cdot6 \ldots (2n+2)} ,$$

and

$$P_{2n-2}(0) = (-1)^{n-1} \frac{1\cdot3\cdot5 \ldots (2n-3)}{2\cdot4\cdot6 \ldots (2n-2)} .$$

Therefore,

$$a_{2n} = (-1)^{n+1} \frac{1\cdot3\cdot5 \ldots (2n+1)}{2\cdot4\cdot6 \ldots (2n+2)(4n+3)} - (-1)^n \frac{1\cdot3\cdot5 \ldots (2n-1)}{2\cdot4\cdot6 \ldots (2n)(4n+3)}$$

$$- (-1)^n \frac{1\cdot3\cdot5 \ldots (2n-1)}{2\cdot4\cdot6 \ldots (2n)(4n-1)} + (-1)^{n-1} \frac{1\cdot3\cdot5 \ldots (2n-3)}{2\cdot4\cdot6 \ldots (2n-2)(4n-1)}$$

$$= (-1)^{n+1} \frac{1\cdot3\cdot5 \ldots (2n-3)}{2\cdot4\cdot6 \ldots (2n-2)} \left[\frac{(2n-1)(2n+1)}{2n(4n+3)(2n+2)} + \frac{1}{(4n-1)} + \frac{(2n-1)}{2n(4n+3)} + \frac{(2n-1)}{2n(4n-1)}\right]$$

$$= (-1)^{n+1} \frac{1\cdot3\cdot5 \ldots (2n-1)}{2\cdot4\cdot6 \ldots (2n)} \frac{(4n+1)(4n-1)(4n+3)}{(4n+3)(4n-1)(2n-1)(2n+2)}$$

$$= (-1)^{n+1} \frac{1\cdot3\cdot5 \ldots (2n-3)(4n+1)}{2\cdot4\cdot6 \ldots (2n)(2n+2)} .$$

Then

$$a_{2n-2} = (-1)^n \frac{1\cdot3\cdot5 \ldots (2n-3)(4n-3)}{2\cdot4\cdot6 \ldots (2n-2)(2n)(2n-3)} ,$$

hence

$$\text{(V-11)} \qquad a_{2n} = (-1)\,\frac{(4n+1)(2n-3)}{(2n+2)(4n-3)}\,a_{2n-2}\,, \qquad n \neq 0.$$

For n = 0, from (V-9)

$$a_0 = [2(0)+1]\int_0^1 x\,P_0(x)\,dx = 1\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}\,,$$

from (V-10), when n = 1

$$a_1 = 0.$$

From (V-11), when n = 2

$$a_2 = +\frac{5}{4}\,a_0 = \frac{5}{4}\,\frac{1}{2} = \frac{5}{8}\,,$$

$$a_3 = 0,$$

$$a_4 = -\frac{3}{16}\,,$$

$$a_5 = 0,$$

$$a_6 = \frac{13}{128}\,,$$

$$\cdots\cdots\cdots\cdots$$

Then

$$f(x) = \sum_{n=0}^{\infty} a_n\,P_n(x), \qquad\qquad -1 < x < 1,$$

$$= \frac{1}{2} + \frac{5}{8}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{3}{16}\left(\frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}\right) + \cdots\,.$$

A graphic representation of the first few terms of the series is shown in Figure 6.

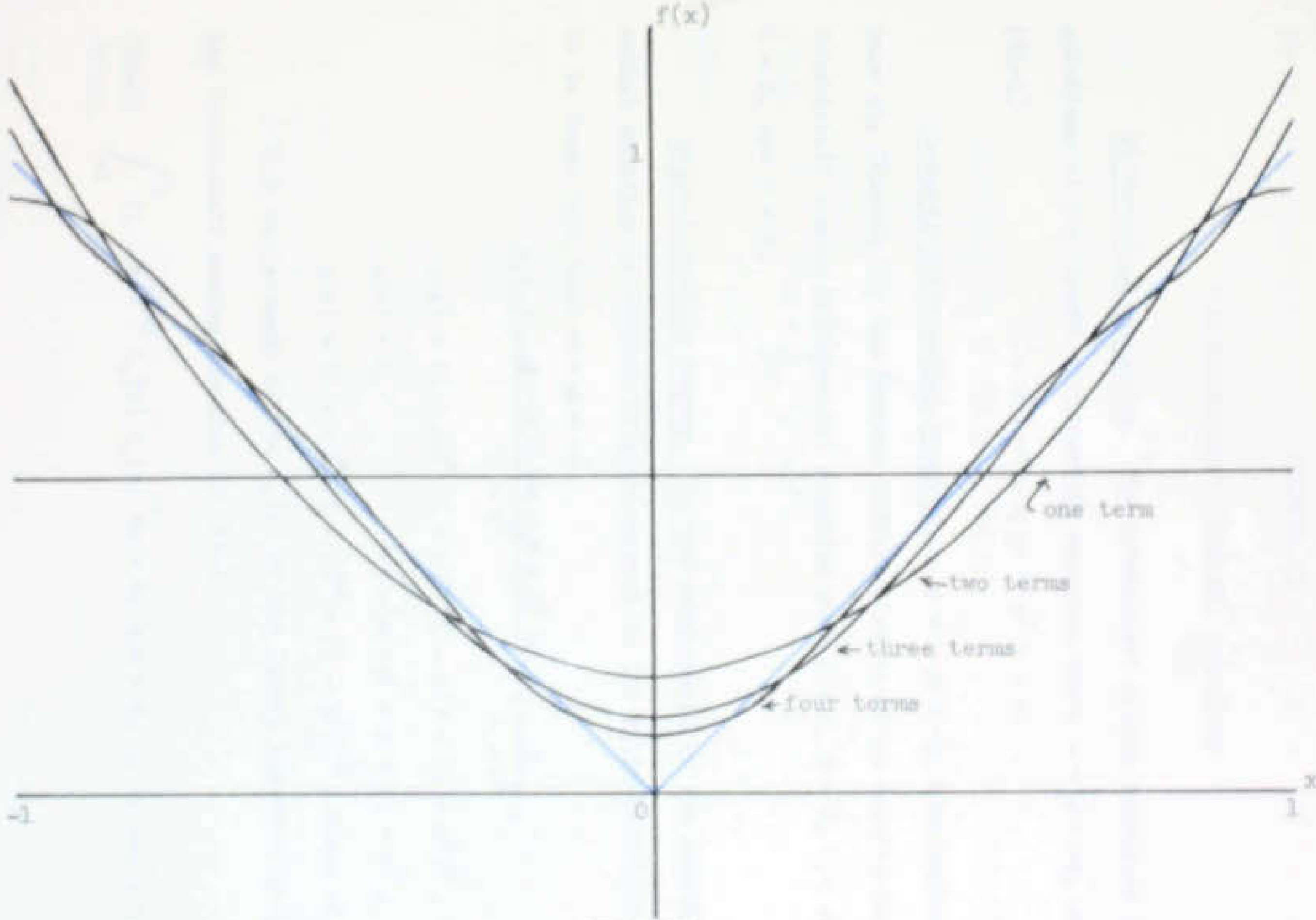

FIGURE 6

A Graphic Representation of the Legendre Series: $f(x) = -x$, $-1 < x < 0$; $f(x) = x$, $0 < x < 1$.

CHAPTER VI

THE TCHEBICHEFF (COSINE) FUNCTIONS

Differential equation. The Tchebicheff cosine functions are solutions of the Jacobi differential equation where $\alpha = \beta = -\frac{1}{2}$, or

$$(1 - x^2)\, y'' - x\, y' + n^2\, y = 0. \tag{VI-1}$$

General differential equation. As $\alpha = \beta = -\frac{1}{2}$, then according to page 27, Chapter IV, the Pearson differential equation results in the Tchebicheff cosine differential equation when A = 1, B = 0, C = -1, D = 0, and E = 1.

Sturm-Liouville theory. In the development of the Jacobi differential equation in Chapter IV with respect to the Sturm-Liouville theory, it is found that when $\alpha = \beta = -\frac{1}{2}$

$$f_1(x) = \frac{\beta - \alpha - (\alpha + \beta + 2)\, x}{1 - x^2} = -\frac{x}{1 - x^2},$$

$$r(x) = (1 + x)^{\beta}(1 - x)^{\alpha}(1 - x^2) = (1 - x^2)^{\frac{1}{2}},$$

$$q(x) = 0, \qquad \lambda = n(\alpha + \beta + n + 1) = n^2,$$

$$p(x) = (1 + x)^{\beta}(1 - x)^{\alpha} = (1 - x^2)^{-\frac{1}{2}}, \text{ where } -1 < x < 1.$$

With replacement of the $J_n(x)$ of the Jacobi function by $C_n(x)$ for the Tchebicheff cosine function in (IV-1)

$$\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}}\, C_m(x)\, C_n(x)\, dx = 0,\ m,n = 0, 1, 2, \ldots, m \neq n. \tag{VI-2}$$

If the C(x)'s are taken to be[1]

$$\cos (n \arccos x), \qquad n = 1, 2, 3, \ldots,$$
$$-1 < x < 1,$$

and $$C_0(x) = 1, \qquad \text{where } n = 0,$$

it is seen that (VI-2) is satisfied.

Determination of the coefficients. Since the Tchebichef cosine functions are special cases of the Jacobi functions when $\alpha = \beta = -\frac{1}{2}$, then by (IV-3) a_n of the series

$$\text{(VI-3)} \qquad f(x) = \sum_{n=0}^{\infty} a_n C_n(x), \qquad -1 < x < 1,$$

equals

$$\frac{1}{\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} C_n^2(x)\, dx} \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} C_n(x) f(x)\, dx,$$

where

$$\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} C_n^2(x)\, dx = \qquad (n = 1, 2, 3, \ldots)$$

$$\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} \cos^2 (n \arccos x)\, dx =$$

$$-\int_{\pi}^{0} \cos^2 n\theta\, d\theta = \int_{0}^{\pi} \cos^2 n\theta\, d\theta = \frac{1}{n} \left[\frac{n\theta}{2} + \frac{\sin 2n\theta}{4} \right]_0^{\pi} = \frac{1}{n} \frac{n\pi}{2} = \frac{\pi}{2},$$

if $$\theta = \arccos x,$$

$$d\theta = - (1 - x^2)^{-\frac{1}{2}}\, dx, \qquad \text{where } \pi < \theta < 0.$$

Hence,

---

[1]Churchill, op. cit., p. 44.

$$a_n = \frac{2}{\pi}\int_{-1}^{1}(1 - x^2)^{-\frac{1}{2}}\, C_n(x)\, f(x)\, dx, \qquad n = 1, 2, 3, \ldots .$$

When $n = 0$,

$$\int_{-1}^{1}(1 - x^2)^{-\frac{1}{2}}\, C_0^2(x)\, dx = \int_{-1}^{1}(1 - x^2)^{-\frac{1}{2}}\, dx = \Big[\arcsin x\Big]_{-1}^{1} = \pi,$$

hence

$$a_0 = \frac{1}{\pi}\int_{-1}^{1}(1 - x^2)^{-\frac{1}{2}}\, f(x)\, dx.$$

<u>Expansion of an arbitrary function in series</u>. Let the arbitrary function be

$$f(x) = 0,\ -1 < x < 0;\ f(x) = 1,\ 0 < x < 1.$$

Then

$$a_0 = \frac{1}{\pi}\int_{-1}^{1}(1 - x^2)^{-\frac{1}{2}}\, f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-1}^{0}(1 - x^2)^{-\frac{1}{2}}\,(0)\, dx + \frac{1}{\pi}\int_{0}^{1}(1 - x^2)^{-\frac{1}{2}}\,(1)\, dx$$

$$= \frac{1}{\pi}\int_{0}^{1}(1 - x^2)^{-\frac{1}{2}}\, dx = \frac{1}{\pi}\Big[\arcsin x\Big]_{0}^{1} = \frac{1}{2},$$

and when $n = 1, 2, 3, \ldots$,

$$a_n = \frac{2}{\pi}\int_{-1}^{1}(1 - x^2)^{-\frac{1}{2}}\, C_n(x)\, f(x)\, dx$$

$$= \frac{2}{\pi}\int_{-1}^{0}(1 - x^2)^{-\frac{1}{2}}\,(0)\cos(n \arccos x)\, dx$$

$$+ \frac{2}{\pi}\int_{0}^{1}(1 - x^2)^{-\frac{1}{2}}\,(1)\cos(n \arccos x)\, dx$$

$$= \frac{2}{\pi}\int_0^1 (1 - x^2)^{-\frac{1}{2}} \cos (n \arccos x)\, dx.$$

Let $\arccos x = \theta$; $-(1 - x^2)^{-\frac{1}{2}}\, dx = d\theta$;

$$a_n = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} - \cos n\theta\, d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos n\theta\, d\theta$$

$$= \frac{2}{n\pi}\Big[\sin n\theta\Big]_0^{\frac{\pi}{2}} = \frac{2}{n\pi} \sin \frac{n\pi}{2}.$$

When n is even, $a_n = 0$; hence

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\, C_n(x)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} a_{2n-1} \cos (2n - 1) \arccos x,$$

$$= \frac{1}{2} + \frac{2}{\pi} \cos (\arccos x) - \frac{2}{3\pi} \cos 3(\arccos x) + \ldots, \quad -1 < x < 1.$$

The graph in Figure 7 is a representation of the first few terms of the above series.

<u>Expansion of a second arbitrary function in series</u>. For a second arbitrary function, let

$$f(x) = -x, \ -1 < x < 0; \ f(x) = x, \ 0 < x < 1.$$

Then

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-1}^{0} (1 - x^2)^{-\frac{1}{2}} (-x)\, dx + \frac{1}{\pi}\int_0^1 (1 - x^2)^{-\frac{1}{2}} (x)\, dx$$

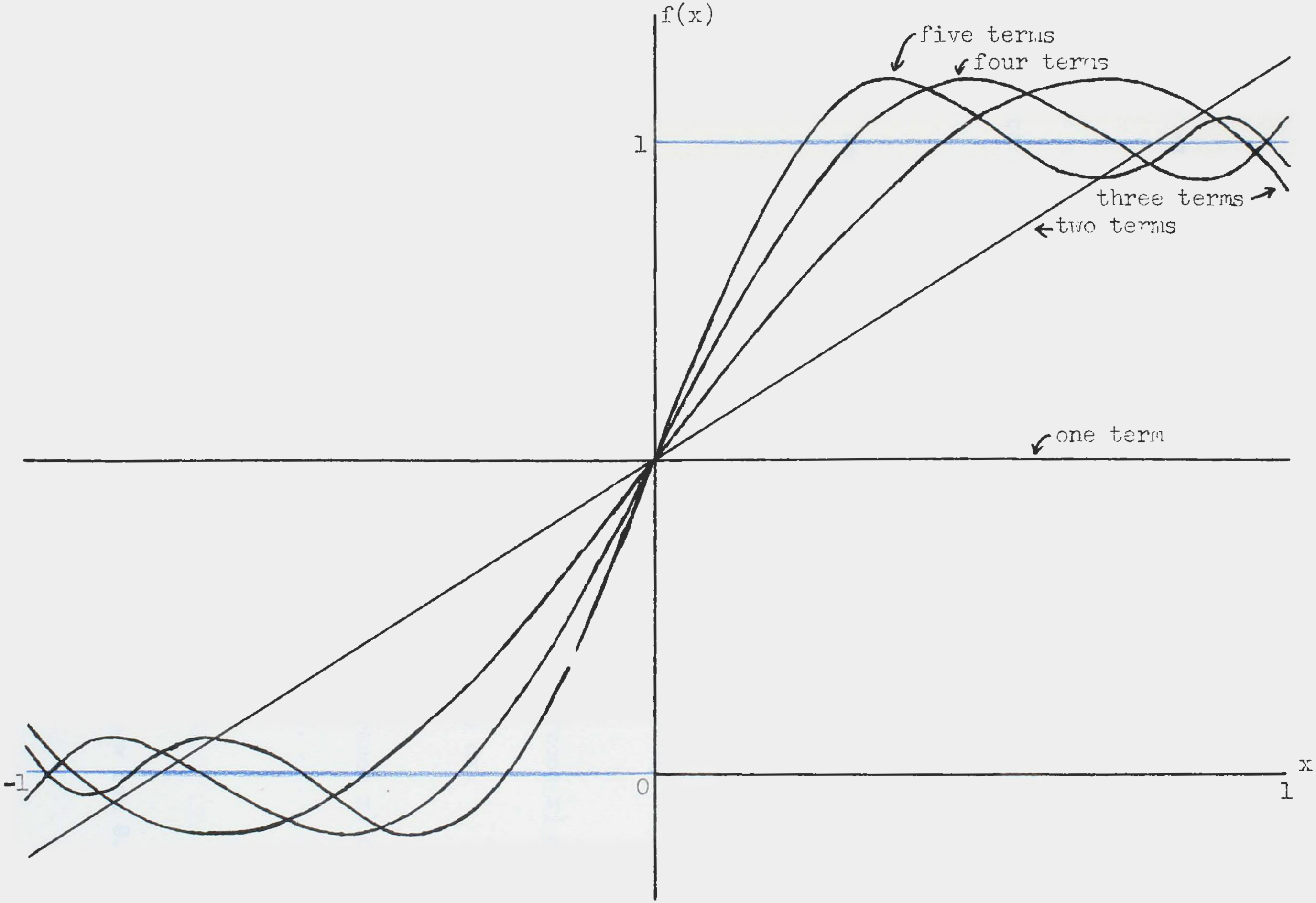

FIGURE 7

A Graphic Representation of the Tchebicheff Cosine Series:
$f(x) = 0, \ -1 < x < 0; \ f(x) = 1, \ 0 < x < 1.$

$$= \frac{1}{\pi}\int_0^{-1} x(1 - x^2)^{-\frac{1}{2}}\, dx + \frac{1}{\pi}\int_0^{1} x(1 - x^2)^{-\frac{1}{2}}\, dx.$$

Let $x = -z$ in the first integral;

$$a_0 = \frac{1}{\pi}\int_0^{1} (-z)(1 - z^2)^{-\frac{1}{2}}\,(-dz) + \frac{1}{\pi}\int_0^{1} x(1 - x^2)^{-\frac{1}{2}}\, dx$$

and since x and z are dummy variables

$$a_0 = \frac{2}{\pi}\int_0^{1} x(1 - x^2)^{-\frac{1}{2}}\, dx = \frac{2}{\pi}\Big[- (1 - x^2)^{\frac{1}{2}}\Big]_0^1 = \frac{2}{\pi}\ .$$

Now where $n = 1, 2, 3, \ldots$

(VI-4) $$a_n = \frac{2}{\pi}\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}}\, C_n(x)\, f(x)\, dx$$

$$= \frac{2}{\pi}\int_{-1}^{0} (1 - x^2)^{-\frac{1}{2}}\,(-x) \cos (n \arccos x)\, dx$$

$$+ \frac{2}{\pi}\int_0^{1} (1 - x^2)^{-\frac{1}{2}}\,(x) \cos (n \arccos x)\, dx$$

$$= \frac{2}{\pi}\int_0^{-1} (1 - x^2)^{-\frac{1}{2}}\,(x) \cos (n \arccos x)\, dx$$

$$+ \frac{2}{\pi}\int_0^{1} (1 - x^2)^{-\frac{1}{2}}\,(x) \cos (n \arccos x)\, dx\ .$$

For odd n, $a_n = 0$. For even n,

$$a_n = \frac{4}{\pi}\int_0^{1} (1 - x^2)^{-\frac{1}{2}}\, x \cos (n \arccos x)\, dx.$$

Let $\arccos x = \theta$, then $- (1 - x^2)^{-\frac{1}{2}}\, dx = d\theta$, and $x = \cos\theta$.

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos\theta \cos n\theta\, d\theta$$

$$= \frac{4}{\pi}\left[\frac{1}{2(n-1)}\sin(n-1)\theta + \frac{1}{2(n+1)}\sin(n+1)\theta\right]_0^{\frac{\pi}{2}}$$

(VI-5) $$= \frac{2}{\pi}\left[\frac{1}{n-1}\sin(n-1)\frac{\pi}{2} + \frac{1}{n+1}\sin(n+1)\frac{\pi}{2}\right].$$

When $n = 3, 5, 7, \ldots, a_n = 0$. It will be noted that when $n = 1$, the expression (VI-5) gives an indeterminate form; hence a_1 must be determined with the use of (VI-4) where $n = 1$,

$$a_1 = \frac{2}{\pi}\int_{-1}^{0}(1-x^2)^{-\frac{1}{2}}(-x)\cos(\arccos x)\,dx$$

$$+ \frac{2}{\pi}\int_{0}^{1}(1-x^2)^{-\frac{1}{2}}(x)\cos(\arccos x)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{-1}(1-x^2)^{-\frac{1}{2}}x^2\,dx + \frac{2}{\pi}\int_{0}^{1}(1-x^2)^{-\frac{1}{2}}x^2\,dx$$

$$= -\frac{2}{\pi}\int_{0}^{1}(1-x^2)^{-\frac{1}{2}}x^2\,dx + \frac{2}{\pi}\int_{0}^{1}(1-x^2)^{-\frac{1}{2}}x^2\,dx = 0.$$

Therefore,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n C_n(x)$$

$$= a_0 + \sum_{n=1}^{\infty} a_{2n}\cos(2n\arccos x), \qquad -1 < x < 1,$$

$$= \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}\sin(2n-1)\frac{\pi}{2} + \frac{1}{2n+1}\sin(2n+1)\frac{\pi}{2}\right]$$

$$\cdot\cos(2n\arccos x)$$

$$= \frac{2}{\pi} + \frac{4}{3\pi}\cos 2(\arccos x) - \frac{4}{15\pi}\cos 4(\arccos x) + \ldots,$$

$$-1 < x < 1.$$

A graphic interpretation of the first few terms of the foregoing series is illustrated in Figure 8.

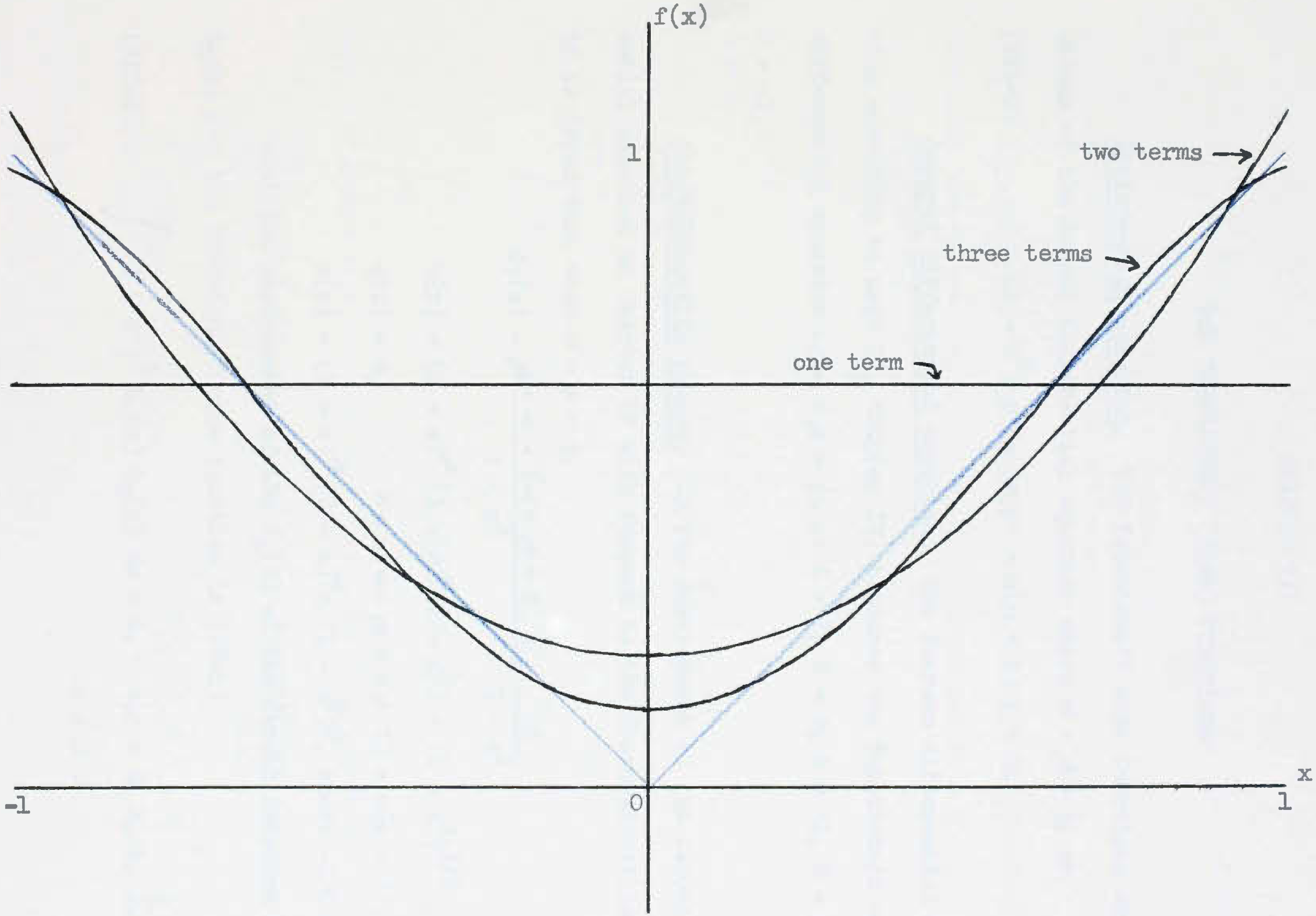

FIGURE 8

A Graphic Representation of the Tchebicheff Cosine Series:
$f(x) = -x, \ -1 < x < 0; \ f(x) = x, \ 0 < x < 1.$

CHAPTER VII

THE TCHEBICHEFF (SINE) FUNCTIONS

Differential equation. The Tchebicheff sine functions are solutions of the Jacobi differential equation where $\alpha = \beta = \frac{1}{2}$, or

$$(1 - x^2)\,y'' - 3x\,y' + n(n + 2)\,y = 0. \tag{VII-1}$$

General differential equation. The Pearson differential equation according to page 27, Chapter IV, produces the Tchebicheff sine differential equation as $\alpha = \beta = \frac{1}{2}$, or $A = 1$, $B = 0$, $C = -1$, $D = 0$, and $E = -1$.

Sturm-Liouville theory. In the development of the Jacobi differential equation in Chapter IV with respect to the Sturm-Liouville theory, it is found that when $\alpha = \beta = \frac{1}{2}$,

$$f_1(x) = \frac{\beta - \alpha - (\alpha + \beta + 2)\,x}{1 - x^2} = \frac{-3}{1 - x^2}\,,$$

$$r(x) = (1 + x)^{\beta}(1 - x)^{\alpha}(1 - x^2) = (1 - x^2)^{3/2}\,,$$

$$q(x) = 0, \qquad \lambda = n(\alpha + \beta + n + 1) = n(n + 2),$$

$$p(x) = (1 + x)^{\beta}(1 - x)^{\alpha} = (1 - x^2)^{\frac{1}{2}}, \text{ where } -1 < x < 1.$$

With the replacement of the $J_n(x)$ of the Jacobi function by $S_n(x)$ for the Tchebicheff sine function in (IV-1)

$$\int_{-1}^{1}(1 - x^2)^{\frac{1}{2}}\,S_m(x)\,S_n(x)\,dx = 0, \qquad m,n = 0, 1, 2, \ldots,\quad m \neq n. \tag{VII-2}$$

If the S(x)'s are taken to be[1]

$$\frac{\sin\left[(n+1)\arccos x\right]}{\sin(\arccos x)}, \quad n = 1, 2, 3, \ldots, \quad -1 < x < 1,$$

and $\qquad S_0(x) = 1, \qquad$ where $n = 0$,

it can be seen that (VII-2) is satisfied.

Determination of the coefficients. Since the Tchebicheff sine functions are special cases of the Jacobi functions when $\alpha = \beta = \frac{1}{2}$, then by (IV-3) a_n of the series

$$\text{(VII-3)} \qquad f(x) = \sum_{n=0}^{\infty} a_n S_n(x), \qquad -1 < x < 1,$$

equals

$$\frac{1}{\int_{-1}^{1}(1-x^2)^{\frac{1}{2}} S_n^2(x)\,dx} \int_{-1}^{1}(1-x^2)^{\frac{1}{2}} S_n(x) f(x)\,dx,$$

where

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}} S_n^2(x)\,dx = \qquad (n = 0, 1, 2, \ldots)$$

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}} \frac{\sin^2 (n+1)\arccos x}{\sin^2 \arccos x}\,dx.$$

Let $\theta = \arccos x$; $x = \cos\theta$; $d\theta = -(1-x^2)^{-\frac{1}{2}}dx$; where $\pi > \theta > 0$, and

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}} S_n^2(x)\,dx = \int_{\pi}^{0} -(1-\cos^2\theta)\frac{\sin^2(n+1)\theta}{\sin^2\theta}\,d\theta =$$

$$\int_{0}^{\pi}\sin^2(n+1)\theta\,d\theta = \frac{1}{n+1}\left[\frac{(n+1)\theta}{2} - \frac{\sin(n+1)\theta}{4}\right]_0^{\pi} = \frac{\pi}{2}.$$

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[1]Jackson, op. cit., p. 150.

Hence

$$a_n = \frac{2}{\pi}\int_{-1}^{1}(1 - x^2)^{\frac{1}{2}}\, S_n(x)\, f(x)\, dx, \qquad n = 0, 1, 2, 3, \ldots .$$

<u>Expansion of an arbitrary function in series.</u> Let the first arbitrary function be of the form

$$f(x) = 0,\ -1 < x < 0;\ f(x) = 1,\ 0 < x < 1.$$

Then

$$a_n = \frac{2}{\pi}\int_{-1}^{1}(1 - x^2)^{\frac{1}{2}}\, S_n(x)\, f(x)\, dx$$

$$(\text{VII-4}) = \frac{2}{\pi}\int_{-1}^{0}(1 - x^2)^{\frac{1}{2}}\,(0)\, S_n(x)\, dx + \frac{2}{\pi}\int_{0}^{1}(1 - x^2)^{\frac{1}{2}}\,(1)\, S_n(x)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{1}(1 - x^2)^{\frac{1}{2}}\, \frac{\sin\left[(n + 1)\arccos x\right]}{\sin(\arccos x)}\, dx.$$

Let $\arccos x = \theta$; $x = \cos\theta$; $-(1 - x^2)^{-\frac{1}{2}}\, dx = d\theta$; where $\pi > \theta > 0$; then

$$a_n = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} - (1 - \cos^2\theta)\, \frac{\sin(n + 1)\theta}{\sin\theta}\, d\theta$$

$$= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\sin\theta\, \sin(n + 1)\theta\, d\theta$$

$$= \frac{2}{\pi}\left[\frac{\sin n\theta}{2n} - \frac{\sin(n + 2)\theta}{2(n + 2)}\right]_0^{\frac{\pi}{2}} = \frac{1}{\pi}\left[\frac{(n + 2)\sin n\frac{\pi}{2} - n\sin(n + 2)\frac{\pi}{2}}{n(n + 2)}\right]$$

$$n = 0, 1, 2, \ldots .$$

However, when $n = 0$, it will be noted that a_n is undefined, and when $n = 2, 4, 6, \ldots$, $a_n = 0$. If n is replaced by $2n - 1$,

$$a_{2n-1} = \frac{1}{\pi} \cdot \frac{(2n+1)\sin(2n-1)\frac{\pi}{2} - (2n-1)\sin(2n+1)\frac{\pi}{2}}{(2n-1)(2n+1)} .$$

Now by using (VII-4) where $n = 0$ and $S_0(x) = 1$

$$a_0 = \frac{2}{\pi}\int_0^1 (1 - x^2)^{\frac{1}{2}}\, dx = \frac{2}{\pi}\,\frac{1}{2}\left[x(1 - x^2)^{\frac{1}{2}} + \arcsin x\right]_0^1 = \frac{1}{2} .$$

Hence

$$f(x) = a_0\, S_0(x) + \sum_{n=1}^{\infty} a_{2n-1}\, S_{2n-1}(x), \qquad -1 < x < 1,$$

$$= \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(2n+1)\sin(2n-1)\frac{\pi}{2} - (2n-1)\sin(2n+1)\frac{\pi}{2}}{(2n-1)(2n+1)}$$

$$\cdot \frac{\sin(2n \arccos x)}{\sin \arccos x}$$

$$= \frac{1}{2} + \frac{4}{3\pi}\,\frac{\sin 2\,(\arccos x)}{\sin(\arccos x)} - \frac{8}{15\pi}\,\frac{\sin 4\,(\arccos x)}{\sin(\arccos x)} + \cdots .$$

A graphic representation of the first few terms of the above series is shown in Figure 9.

Expansion of a second arbitrary function in series. For a second arbitrary function, let

$$f(x) = -x, \; -1 < x < 0; \; f(x) = x, \; 0 < x < 1.$$

Then

$$a_n = \frac{2}{\pi}\int_{-1}^{1} (1 - x^2)^{\frac{1}{2}}\, S_n(x)\, f(x)\, dx$$

$$= \frac{2}{\pi}\int_{-1}^{0} (1 - x^2)^{\frac{1}{2}}\, (-x)\, S_n(x)\, dx + \frac{2}{\pi}\int_{0}^{1} (1 - x^2)^{\frac{1}{2}}\, (x)\, S_n(x)\, dx$$

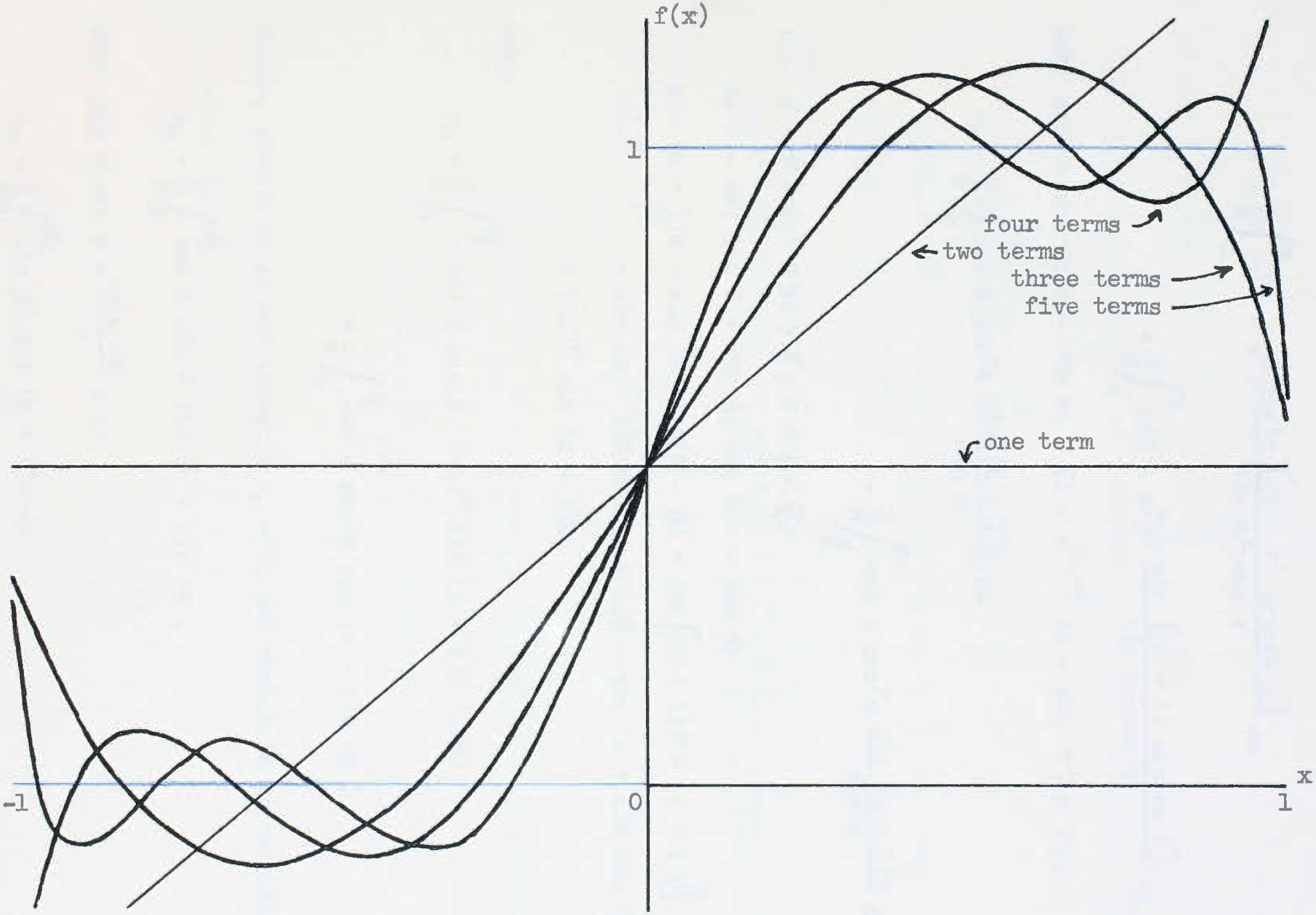

FIGURE 9

A Graphic Representation of the Tchebicheff Sine Series:
$f(x) = 0, -1 < x < 0; f(x) = 1, 0 < x < 1.$

$$= -\frac{2}{\pi}\int_{-1}^{0} x(1-x^2)^{\frac{1}{2}} \frac{\sin\left[(n+1)\arccos x\right]}{\sin \arccos x}\, dx$$

$$+\frac{2}{\pi}\int_{0}^{1} x(1-x^2)^{\frac{1}{2}} \frac{\sin\left[(n+1)\arccos x\right]}{\sin \arccos x}\, dx\ .$$

Let $\arccos x = \theta$; $x = \cos\theta$; $-(1-x^2)^{-\frac{1}{2}}\, dx = d\theta$; $\pi > \theta > 0$,

$$a_n = \frac{2}{\pi}\int_{\pi}^{\frac{\pi}{2}} \cos\theta \sin^2\theta \frac{\sin(n+1)\theta}{\sin\theta}\, d\theta$$

$$-\frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} \cos\theta \sin^2\theta \frac{\sin(n+1)\theta}{\sin\theta}\, d\theta\ .$$

Let $\theta = \pi - \phi$; $\pi > \theta > \frac{\pi}{2}$; $0 < \phi < \frac{\pi}{2}$;

$d\theta = -d\phi$; $\sin\theta = \sin\phi$; $\cos\theta = -\cos\phi$;

$$\sin(n+1)\theta = \sin(n+1)(\pi-\phi) = \sin\left[(n+1)\pi - (n+1)\phi\right]$$

$$= \sin(n+1)\pi \cos(n+1)\phi - \cos(n+1)\pi \sin(n+1)\phi$$

$$= (-1)^n \sin(n+1)\phi,$$

then

$$a_n = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} -\cos\phi \sin\phi \ (-1)^n \sin(n+1)\phi\ (-d\phi)$$

$$+\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos\theta \sin\theta \sin(n+1)\theta\, d\theta\ .$$

Hence, when n is an odd integer, $a_n = 0$; and when n is an even integer,

$$a_n = \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} \cos\theta \sin\theta \sin(n+1)\theta\, d\theta\ .$$

Now $\cos\theta \sin\theta = \dfrac{\sin 2\theta}{2}$; so

$$a_n = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \sin 2\theta \sin(n+1)\theta\, d\theta$$

$$= \frac{2}{\pi}\left[\frac{\sin (n - 1)\theta}{2(n - 1)} - \frac{\sin (n + 3)\theta}{2(n - 1)}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{\sin (n - 1)\frac{\pi}{2}}{n - 1} - \frac{\sin (n + 3)\frac{\pi}{2}}{n + 3}\right].$$

Since n is an even integer, substitution of 2k for n, where k = 0, 1, 2, 3, ... , gives

$$a_{2k} = \frac{1}{\pi}\left[\frac{\sin (2k - 1)\frac{\pi}{2}}{(2k - 1)} - \frac{\sin (2k + 3)\frac{\pi}{2}}{(2k * 3)}\right],$$

but $\sin (2k - 1)\frac{\pi}{2} = \sin (2k + 3)\frac{\pi}{2}$, k = 0, 1, 2, 3, ... , and

$\sin (2k - 1)\frac{\pi}{2} = (-1)^{k+1}$;

then

$$a_{2k} = \frac{(-1)^{k+1}}{\pi}\left[\frac{1}{2k - 1} - \frac{1}{2k + 3}\right] = (-1)^{k+1}\frac{4}{\pi}\left[\frac{1}{(2k - 1)(2k + 3)}\right].$$

Hence

$$f(x) = \sum_{k=0}^{\infty} a_{2k}\, S_{2k}(x)$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{4}{(2k - 1)(2k + 3)} \frac{\sin \left[(2k + 1) \arccos x\right]}{\sin \arccos x},$$

$$= \frac{4}{3\pi} + \frac{4}{5\pi}\frac{\sin 3(\arccos x)}{\sin \arccos x} - \frac{4}{21\pi}\frac{\sin 5(\arccos x)}{\sin \arccos x} + \dots ,$$

$$-1 < x < 1.$$

In Figure 10 a graphic illustration of the first few terms of the above series can be seen.

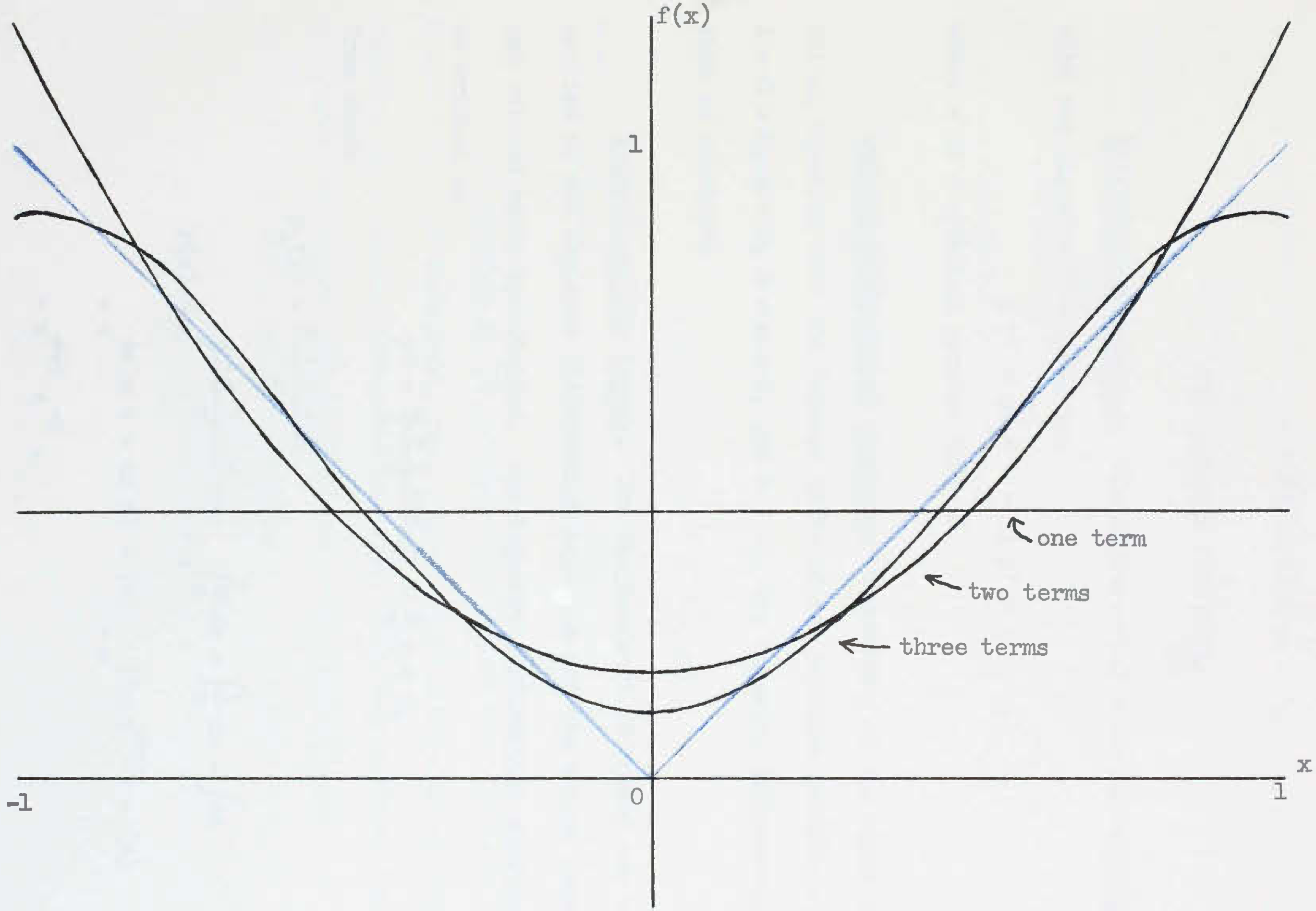

FIGURE 10

A Graphic Representation of the Tchebicheff Sine Series:
$f(x) = -x, \ -1 < x < 0; \ f(x) = x, \ 0 < x < 1.$

CHAPTER VIII

THE LAGUERRE FUNCTIONS

<u>Differential</u> <u>equation</u>. The differential equation associated with the Laguerre functions is:

$$x\, y'' + (\alpha + 1 - x)\, y' + n\, y = 0,$$

where α is a constant greater than -1.

<u>General</u> <u>differential</u> <u>equation</u>. Comparison of the above differential equation with the Pearson differential equation reveals that when $A = C = 0$, $B = 1$, $D = \alpha + 1$, and $E = -1$, the Laguerre differential equation is obtained.

<u>Sturm-Liouville</u> <u>theory</u>. The Sturm-Liouville theory can be applied to the Laguerre differential equation and the weight function and related parts ascertained. The Laguerre differential equation can be written as

$$y'' + \frac{\alpha + 1 - x}{x}\, y' + \frac{n}{x}\, y = 0,$$

from which

$$f_1(x) = \frac{\alpha + 1 - x}{x}$$

$$r(x) = e^{\int \frac{\alpha + 1 - x}{x}\, dx} = e^{\int \frac{\alpha}{x}\, dx + \int \frac{1}{x}\, dx - \int dx}$$

$$= e^{(\alpha \ln x + \ln x) - (x)} = e^{(\ln x^{\alpha+1}) - (x)}$$

$$= x^{\alpha+1}\, e^{-x} .$$

The Laguerre differential equation can be written as

$$\frac{d}{dx}\left[x^{\alpha+1}\, e^{-x}\, \frac{dy}{dx}\right] + n\, x^{\alpha} e^{-x}\, y = 0,$$

from which

$$q(x) = 0; \quad \lambda = n; \quad p(x) = x^{\alpha}\, e^{-x}.$$

The weight function, $p(x)$, is $x^{\alpha} e^{-x}$; so if the Laguerre functions are represented by $L_n(x)$, $n = 0, 1, 2, \ldots$, they will be orthogonal as

(VIII-1) $$\int_0^{\infty} x^{\alpha}\, e^{-x}\, L_m(x)\, L_n(x)\, dx = 0, \quad m,n = 0, 1, 2, \ldots, \quad m \neq n.$$

The functions are determined[1] to be

(VIII-2) $$L_n(x) = (-1)^n\, x^{-\alpha}\, e^{x}\, \frac{d^n}{dx^n}\, \phi_n(x),$$

where

$$\phi_n(x) = x^{\alpha+n}\, e^{-x}.$$

<u>Determination of the coefficients</u>. Since the various L(x)'s are determined to be orthogonal, then without great difficulty an arbitrary function of x can be represented in series form that utilizes these orthogonal functions. This function of x can be written as

(VIII-3) $$f(x) = a_0 L_0(x) + a_1 L_1(x) + \ldots + a_n L_n(x) + \ldots$$

$$= \sum_{n=0}^{\infty} a_n\, L_n(x), \qquad 0 < x < \infty.$$

---

[1] Jackson, <u>op</u>. <u>cit</u>., p. 184.

Multiplication of (VIII-3) by $L_n(x)$ and the weight function $p(x) = x^{\alpha} e^{-x}$, and integration of the resulting expression $x^{\alpha} e^{-x} f(x) L_n(x)$ from 0 to ∞, integration term by term assumed to be legitimate, reduce each integral on the right to zero, by (VIII-1), except the integral containing $L_n^2(x)$, and give

$$\int_0^{\infty} x^{\alpha} e^{-x} L_n(x) f(x)\, dx = a_n \int_0^{\infty} x^{\alpha} e^{-x} L_n^2(x)\, dx.$$

However,

$$\int_0^{\infty} x^{\alpha} e^{-x} L_n^2(x)\, dx = n!\,\Gamma(\alpha + n + 1);$$ [2]

so

$$\text{(VIII-4)} \qquad a_n = \frac{1}{n!\,\Gamma(\alpha + n + 1)} \int_0^{\infty} x^{\alpha} e^{-x} L_n(x) f(x)\, dx, \quad \alpha > -1.$$

<u>Expansion of an arbitrary function</u>. Let the first arbitrary function be

$$f(x) = 1, \; 0 < x < \infty.$$

Since the Laguerre functions are more restrictively associated with the sime exponential weight function e^{-x},[3] the arbitrary functions will be determined by utilizing $\alpha = 0$ in the foregoing equations. Then a_n of (VIII-3) will equal

$$a_n = \frac{1}{n!\,\Gamma(n + 1)} \int_0^{\infty} e^{-x} L_n(x)\, dx, \qquad n = 0, 1, 2, \ldots .$$

---

[2] <u>Ibid</u>., p. 185.

[3] Ibid., p. 184.

Substitution of (VIII-2) in the foregoing equation gives

$$\text{(VIII-5)} \qquad a_n = \frac{(-1)^n}{n!\,\Gamma(n+1)} \int_0^{\infty} \frac{d^n\, \phi_n(x)}{dx^n}\, dx$$

$$= \frac{(-1)^n}{n!\,\Gamma(n+1)} \left[\frac{d^{n-1}\, \phi_n(x)}{dx^{n-1}}\right]_0^{\infty}, \qquad n \geqslant 1,$$

$$= \frac{(-1)^n}{n!\,\Gamma(n+1)} \left[\frac{d^{n-1}\,(x^n e^{-x})}{dx^{n-1}}\right]_0^{\infty}.$$

For the upper limit after $(n - 1)$ differentiations there will be at least an x factor and always an e^{-x} factor; hence

$$\lim_{x \to \infty} \frac{x^m}{e^x} = 0,$$

and likewise for the lower limit,

$$\lim_{x \to 0} \frac{x^m}{e^x} = 0;$$

then $a_n = 0$, $n \geqslant 1$. It is then left to evaluate a_n where $n = 0$. Substitution of $n = 0$ in (VIII-5) gives

$$a_0 = \frac{1}{0!\,\Gamma(1)} \int_0^{\infty} \phi_0(x)\, dx = \int_0^{\infty} e^{-x}\, dx = \left[- e^{-x}\right]_0^{\infty} = 1.$$

Then

$$f(x) = \sum_{n=0}^{\infty} a_n\, L_n(x), \qquad 0 < x < \infty,$$

$$= L_0(x).$$

By (VIII-4), where $\alpha = 0$, $L_n(x)$, $n = 0, 1, 2, \ldots$, are found to be

$$L_0(x) = 1,$$

$$L_1(x) = x - 1,$$

$$L_2(x) = x^2 - 4x + 2,$$

$$L_3(x) = x^3 - 9x^2 + 18x - 6,$$

$$L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24,$$

$$L_5(x) = x^5 - 25x^4 + 200x^3 - 600x^2 + 600x - 120,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Whence,

$$f(x) = 1, \qquad\qquad 0 < x < \infty.$$

The graphic representation of the above function is shown in Figure 11.

Expansion of a second arbitrary function in series. For the second arbitrary function, let

$$f(x) = x, \qquad\qquad 0 < x < \infty.$$

When = 0 in (VIII-4)

$$a_n = \frac{1}{n!\,\Gamma(n+1)} \int_0^\infty e^{-x} f(x)\, L_n(x)\, dx$$

$$= \frac{1}{n!\,\Gamma(n+1)} \int_0^\infty x\, e^{-x}\, L_n(x)\, dx\ .$$

By (VIII-2)

$$a_n = \frac{(-1)^n}{n!\,\Gamma(n+1)} \int_0^\infty x\, \frac{d^n\, \phi n(x)}{dx^n}\, dx\ .$$

By parts

$$u = x, \qquad\qquad du = dx,$$

$$dv = \frac{d^n\, \phi n(x)}{dx^n}, \qquad\qquad v = \frac{d^{n-1}\, \phi_n(x)}{dx^{n-1}},$$

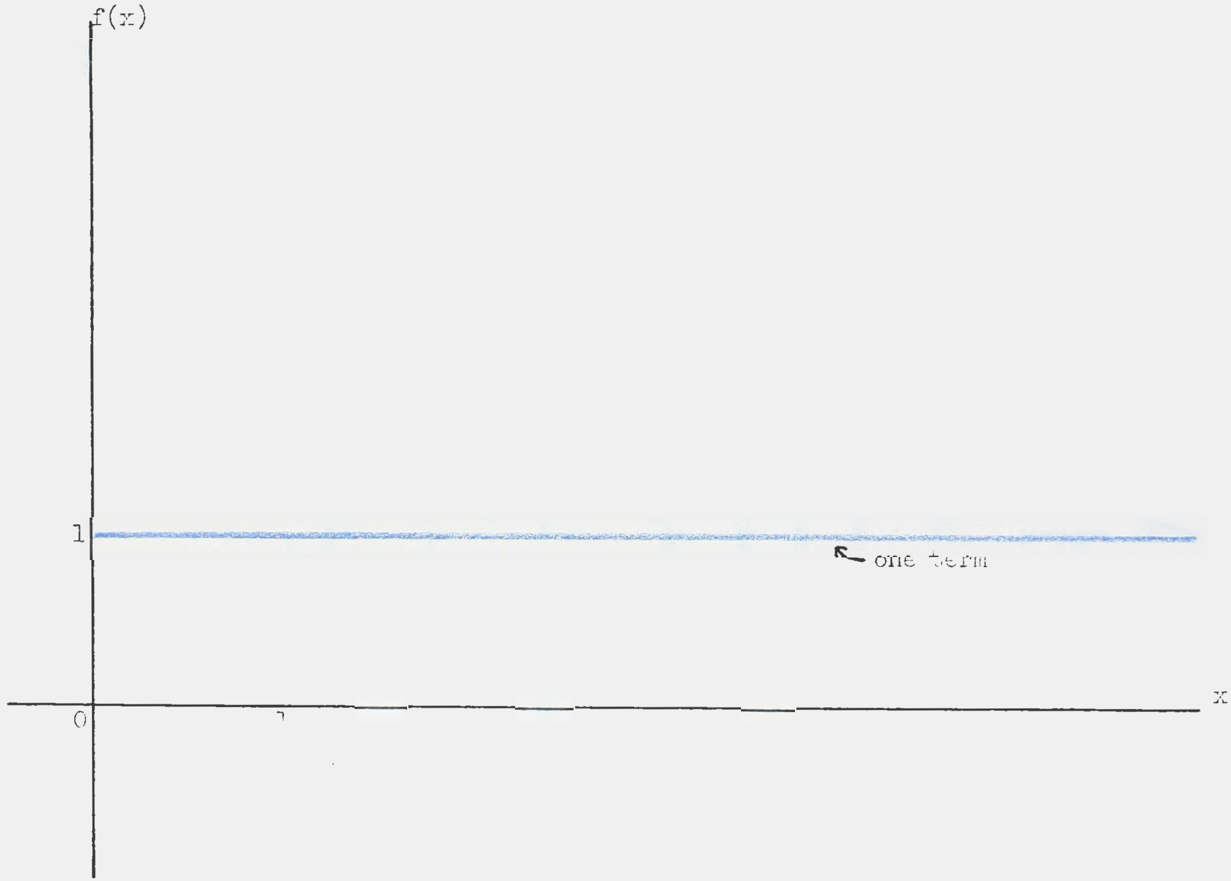

FIGURE 11

Graphic representation of the Laguerre series: $f(x) = 1,\ 0 < x < \infty$.

then since $\Gamma(n + 1) = n!$,

$$a_n = \frac{(-1)^n}{(n!)^2}\left\{\left[x\,\frac{d^{n-1}\,\phi_n(x)}{dx^{n-1}}\right]_0^\infty - \int_0^\infty \frac{d^{n-1}\,\phi_n(x)}{dx^{n-1}}\,dx\right\}$$

$$= \frac{(-1)^n}{(n!)^2}\left\{\left[x\,\frac{d^{n-1}\,\phi_n(x)}{dx^{n-1}}\right]_0^\infty - \left[\frac{d^{n-2}\,\phi_n(x)}{dx^{n-2}}\right]_0^\infty\right\}$$

$$= \frac{(-1)^n}{(n!)^2}\left\{\left[0\right] - \left[0\right]\right\} = 0, \qquad n \geqslant 2,$$

by the utilization of the evaluation on page 60.

It is then left to evaluate a_n when $n = 0$ and $n = 1$. Substitution of $n = 0$ and then $n = 1$ in (VIII-5) gives

$$a_0 = \frac{1}{0!\;1}\int_0^\infty x\,e^{-x}\,(1)\,dx = \left[e^{-x}\,(-x - 1)\right]_0^\infty = 1,$$

$$a_1 = \frac{1}{1!\;1}\int_0^\infty x\,e^{-x}\,(x - 1)\,dx = \int_0^\infty x^2\,e^{-x}\,dx - \int_0^\infty x\,e^{-x}\,dx$$

$$= \left[-x^2\,e^{-x}\right]_0^\infty + 2\int_0^\infty x\,e^{-x}\,dx - \int_0^\infty x\,e^{-x}\,dx$$

$$= \int_0^\infty x\,e^{-x}\,dx = \left[-x\,e^{-x} - e^{-x}\right]_0^\infty = 1.$$

Then

$$f(x) = \sum_{n=0}^{\infty} a_n\,L_n(x), \qquad 0 < x < \infty,$$

$$= a_0\,L_0(x) + a_1\,L_1(x)$$

$$= 1(1) + 1(x - 1) = x, \qquad 0 < x\ \infty.$$

Figure 12 illustrates the graphic representation of the above function.

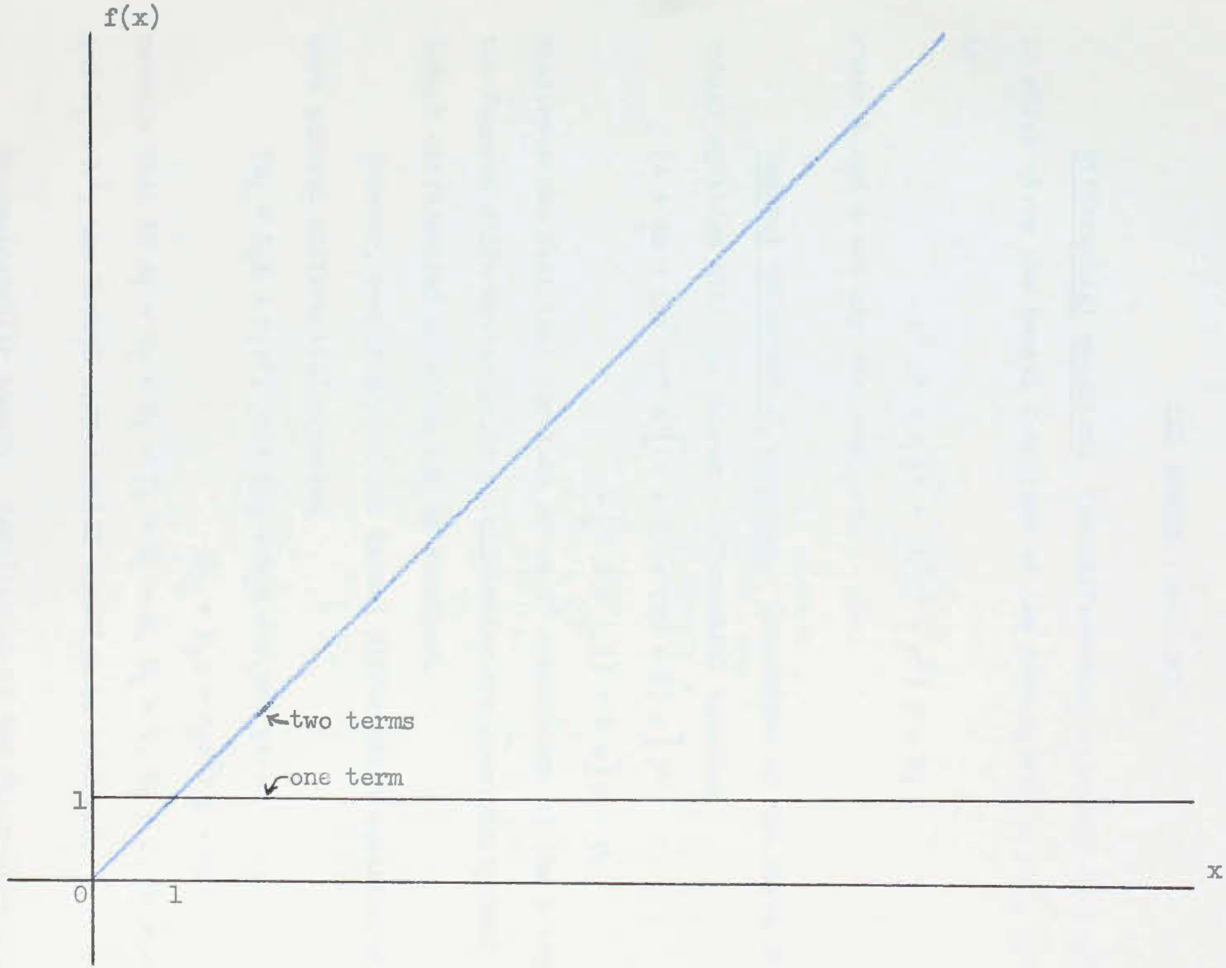

FIGURE 12

A Graphic Representation of the Laguerre Series: $f(x) = x$, $0 < x < \infty$.

CHAPTER IX

THE BESSEL FUNCTIONS

Differential equation. The differential equation, the solution of which gives the Bessel functions of the form $J_n(\lambda x)$, $n = 0, 1, 2, \ldots$ is

$$x^2 y'' + x y' + (\lambda^2 x^2 - n^2) y = 0,$$

where λ and n are any real numbers.

General differential equation. Comparison of the above differential equation with the Pearson differential equation

$$(A + Bx + Cx^2) y'' + \left[(B + D) + (2C + E) x\right] y' - \left[C n(n + 1) + E n\right] y = 0,$$

discloses the fact that the lack of an x^2 coefficient of the y term in the Pearson differential equation eliminates the possibility that the Bessel differential equation can be obtained.

However, comparison of the Bessel differential equation with the more general differential equation

$$(A_1 + B_1 x + C_1 x^2) y'' + (D_1 + E_1 x + F_1 x^2) y' (G_1 + H_1 x + J_1 x^2) y = 0,$$

reveals that if $A_1 = B_1 = E_1 = F_1 = H_1 = 0$, $C_1 = 1$, $E_1 = 1$, $G_1 = -n^2$, and $J_1 = \lambda^2$, the Bessel differential equation is produced.

Sturm-Liouville theory. Application of the Sturm-Liouville theory to the Bessel differential equation in order to ascertain the weight function and related parts discloses that

$$f_1(x) = \frac{1}{x}\text{, and}$$

$$r(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x\ .$$

Hence,

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + (\lambda^2 x - \frac{n^2}{x})\, y = 0,$$

$$q(x) = 0, \quad \lambda_s = \lambda^2, \quad p(x) = x\ .$$

If the Bessel functions are denoted by $J_n(x) = 0$, $n = 0, 1, 2, \ldots$, which has infinitely many positive roots $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, whose values can be calculated to any degree of accuracy. It is established[1] that the functions

$$\sqrt{x}\ J_n(\lambda_1 x),\ \sqrt{x}\ J_n(\lambda_2 x),\ \ldots,\ \sqrt{x}\ J_n(\lambda_k x),\ \ldots,$$

are orthogonal in the interval from $x = 0$ to $x = 1$, so that

$$\text{(IX-1)} \quad \int_0^1 \sqrt{x}\, J_n(\lambda_i x) \cdot \sqrt{x}\, J_n(\lambda_j x)\, dx = 0, \qquad i \neq j,$$

$$= \frac{1}{2}\left[J_n'(\lambda_i)\right]^2, \quad i = j\ .$$

The proof of the above depends on the following identity,

$$\text{(IX-2)} \quad (\lambda^2 - \mu^2)\int_0^x x\, J_n(\lambda x)\, J_n(\mu x)\, dx =$$

$$x\left[\mu J_n(\lambda x)\, J_n'(\mu x) - \lambda J_n(\mu x)\, J_n'(\lambda x)\right].$$

Let $\lambda = \lambda_j$ and $\mu = \lambda_i$, where $\lambda_i \neq \lambda_j$; then

---

[1]Sokolnikoff, op. cit., pp. 339-340.

$$\int_0^x \sqrt{x}\, J_n(\lambda_j x) \cdot \sqrt{x}\, J_n(\lambda_i x)\, dx =$$

$$\frac{x}{\lambda_j^2 - \lambda_i^2}\left[\lambda_i\, J_n(\lambda_j x)\, J_n'(\lambda_i x) - \lambda_j\, J_n(\lambda_i x)\, J_n'(\lambda_j x)\right].$$

If $x = 1$ and since $J_n(\lambda_i) = J_n(\lambda_j) = 0$, then the first part of formula (IX-1) is given.

In order to establish the second part, (IX-2) is differentiated partially with respect to λ, and

$$2\lambda \int_0^x x\, J_n(\lambda x)\, J_n(\mu x)\, dx + (\lambda^2 - \mu^2)\int_0^x x^2\, J_n(\mu x)\, J_n'(\lambda x)\, dx$$

$$= x\left[\mu x\, J_n'(\lambda x)\, J_n'(\mu x) - J_n(\mu x)\, J_n'(\lambda x) - \lambda x\, J_n(\mu x)\, J_n''(\lambda x)\right]$$

is obtained. Set $x = 1$, $\lambda = \mu$, and recall that if λ is a root of $J_n(x) = 0$, then $J_n(\lambda) = 0$,

$$2\lambda \int_0^1 x\, J_n(\lambda x)^2\, dx = -(\lambda^2 - \lambda^2)\int_0^1 x^2\, J_n(\lambda x)\, J_n'(\lambda x)\, dx$$

$$+\left[\lambda\, J_n'(\lambda)\, J_n'(\lambda) - J_n(\lambda)\, J_n'(\lambda) - \lambda\, J_n(\lambda)\, J_n''(\lambda)\right] = \lambda\, J_n'(\lambda)^2,$$

whence

$$\int_0^1 x\, J_n(\lambda x)^2\, dx = \frac{\lambda}{2\lambda} J_n'(\lambda)^2 = \frac{1}{2} J_n'(\lambda)^2.$$

<u>Determination of the coefficients</u>. Since the various Bessel functions are determined to be orthogonal, then it is with little difficulty that a given function of x defined in the interval of orthogonality can be represented. The function of x can be written as

(IX-3) $\quad f(x) = a_1\, J_n(\lambda_1 x) + a_2\, J_n(\lambda_2 x) + \dots + a_k\, J_n(\lambda_k x) + \dots$

$$= \sum_{k=1}^{\infty} a_k J_n(\lambda_k x), \qquad 0 < x < 1,$$

where

$$\text{(IX-4)} \quad J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x)^{n+2k}}{2^{n+2k} \, k! \, \Gamma(n+k+1)}, \qquad n \geqslant 0.$$[2]

If n = 0, the Bessel functions of order zero, which are even functions, are obtained and when n = 1, the Bessel functions of order one, which are odd functions, are obtained. Then since the recursion formula,[3]

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x),$$

gives the function $J_{n+1}(x)$ of any order in terms of the functions $J_n(x)$ and $J_{n-1}(x)$ of lower orders, it is possible to reduce any Bessel function to those of order zero or order one.

Now multiplication of (IX-3) by the weight function $p(x) = x$, and $J_n(\lambda_k x)$, and integration of the resulting expression $x\, f(x)\, J_n(\lambda_k x)$ from 0 to 1, integration term by term assumed to be legitimate, reduce each integral on the right to zero, by (IX-1), except the integral containing $J_n(\lambda_k x)^2$, and give

$$\int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx = a_k \int_0^1 x\, J_n(\lambda_k x)^2\, dx.$$

Then by the second part of (IX-1)

$$a_k = \frac{2}{\left[J_n'(\lambda_k)\right]^2} \int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx, \qquad k = 0, 1, 2, \ldots .$$

---

[2] Ibid., p. 336.

[3] Churchill, op. cit., p. 148.

<u>Expansion of an arbitrary function</u>. Let the first arbitrary function be

$$f(x) = 1, \qquad 0 < x < 1.$$

Since $f(x) = 1$ is an even function and $n = 0$ in (IX-4) gives an even function of $J_n(x)$, then

$$a_k = \frac{2}{\left[J_n'(\lambda_k)\right]^2}\int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx$$

$$= \frac{2}{\left[J_0'(\lambda_k)\right]^2}\int_0^1 x\, J_0(\lambda_k x)\, dx, \qquad k = 1, 2, 3, \ldots .$$

However,

$$J_0'(\lambda_k) = -\, J_1(\lambda_k) \; ,$$ [4]

so

$$a_k = \frac{2}{\left[J_1(\lambda_k)\right]^2}\int_0^1 x\, J_0(\lambda_k x)\, dx \; .$$

But,

$$\int_0^x r\, J_0(r)\, dr = x\, J_1(x) \; ;$$ [5]

so

$$\int_0^1 \lambda_k x\, J_0(\lambda_k x)\, d(\lambda_k x) = \lambda_k\, J_1(\lambda_k).$$

Hence,

$$a_k = \frac{2\, \lambda_k\, J_1(\lambda_k)}{\lambda_k^2 \left[J_1(\lambda_k)\right]^2} = \frac{2}{\lambda_k\, J_1(\lambda_k)} \; .$$

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[4] Sokolnikoff, <u>op</u>. <u>cit</u>., p. 338.

[5] Churchill, <u>op</u>. <u>cit</u>., p. 149.

Then

$$f(x) = \sum_{k=1}^{\infty} a_k\, J_0(\lambda_k x), \qquad 0 < x < 1,$$

$$= 2 \sum_{k=1}^{\infty} \frac{J_0(\lambda_k x)}{\lambda_k\, J_1(\lambda_k)}, \qquad 0 < x < 1.$$

Reference to Table I,[6] which gives the roots of $J_0(x) = 0$ and the corresponding values of $J_1(x)$, gives

$$f(x) = \frac{2\, J_0(2.4048\, x)}{(2.4048)(0.5191)} + \frac{2\, J_0(5.5201\, x)}{(5.5201)(-0.3403)}$$

$$+ \frac{2\, J_0(8.6537\, x)}{(8.6537)(0.2715)} + \frac{2\, J_0(11.7915\, x)}{(11.7915)(-0.2325)} + \cdots,$$

where

$$J_0(\lambda_k x) = 1 - \frac{(\lambda_k\, x)^2}{2^2} + \frac{(\lambda_k\, x)^4}{2^4\, (2!)^2} - \cdots + \frac{(-1)^n\, (\lambda_k\, x)^{2n}}{2^{2n}\, (n!)^2},$$

$$n = 0, 1, 2, \ldots .$$

The graphic representation in Figure 13 shows the first few terms of the above series.

Expansion of a second arbitrary function in series. For the second arbitrary function, let

$$f(x) = x, \qquad 0 < x < 1.$$

Since $f(x) = x$ is an odd function and $n = 1$ in (IX-4) gives an odd function of $J_n(x)$; then,

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[6] Jahnke, Eugene and Fritz Emde, Funktionentafeln, New York, 1943, p. 166.

n	x_n	$J_1(x_n)$
1	2.4048	+ 0.5191
2	5.5201	- 0.3403
3	8.6537	+ 0.2715
4	11.7915	- 0.2325
5	14.9309	+ 0.2065
6	18.0711	- 0.1877
7	21.2116	+ 0.1733
8	24.3525	- 0.1617
9	27.4935	+ 0.1522
10	30.6346	- 0.1442

TABLE I

Roots of $J_0(x) = 0$ and the corresponding values of $J_1(x)$

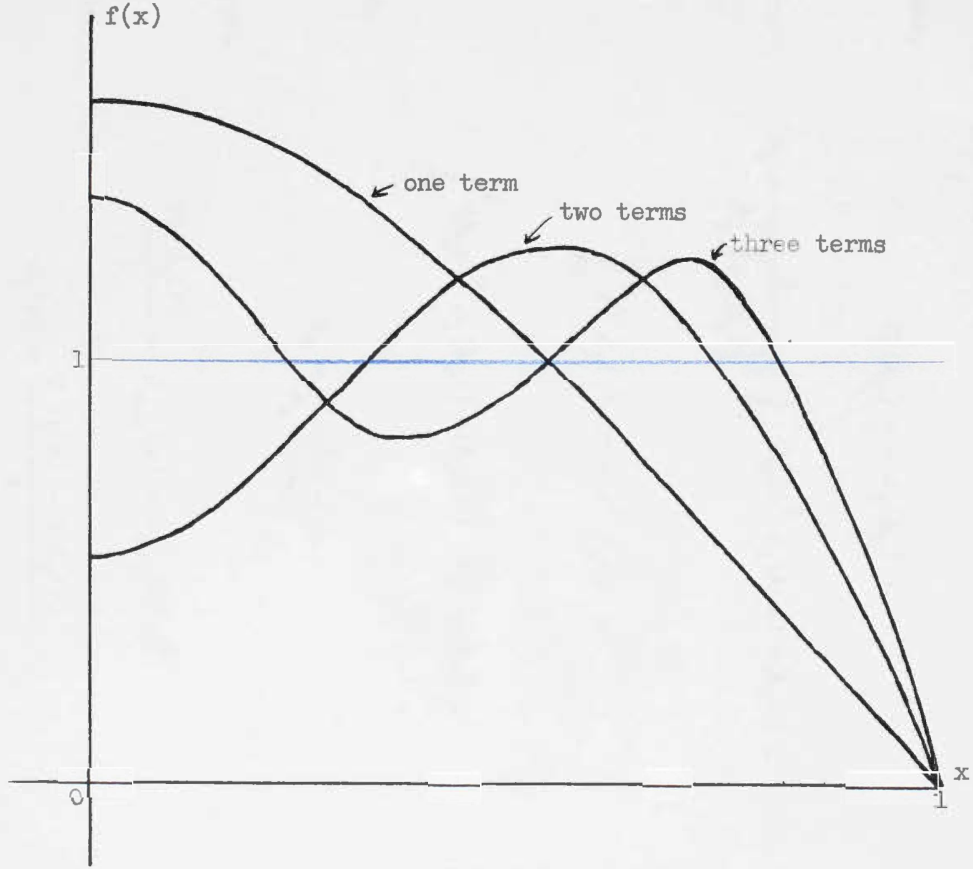

FIGURE 13

A Graphic Representation of the Bessel Series: $f(x) = 1, 0 < x < 1.$

$$a_k = \frac{2}{\left[J_n'(\lambda_k)\right]^2} \int_0^1 x\, f(x)\, J_n(\lambda_k x)\, dx$$

$$= \frac{2}{\left[J_1'(\lambda_k)\right]^2} \int_0^1 x^2\, J_1(\lambda_k x)\, dx, \qquad k = 1, 2, 3, \ldots .$$

However,

$$J_1'(\lambda_k) = -\, J_2(\lambda_k) \;;$$

so

$$a_k = \frac{2}{\lambda_k^3 \left[J_2(\lambda_k)\right]^2} \int_0^1 (\lambda_k x)^2\, J_1(\lambda_k x)\, d(\lambda_k x) .$$

But

$$\int_0^x r^2\, J_1(r)\, dr = x^2\, J_2(x) \;;$$ [7]

so

$$\int_0^1 (\lambda_k x)^2\, J_1(\lambda_k x)\, d(\lambda_k x) = \lambda_k^2\, J_2(\lambda_k) .$$

Hence,

$$a_k = \frac{2}{\lambda_k\, J_2(\lambda_k)} .$$

However,

$$\frac{2n\, J_n(x)}{x} = J_{n-1}(x) + J_{n+1}(x) \;;$$ [8]

so when n = 2,

$$J_2(x) = \frac{2\, J_1(x) - x\, J_0(x)}{x} .$$

---

[7] Churchill, op. cit., p. 148.

[8] Ibid.

Hence

$$a_k = \frac{2}{2\, J_1(\lambda_k) - \lambda_k\, J_0(\lambda_k)} .$$

Then

$$f(x) = \sum_{k=1}^{\infty} a_k\, J_1(\lambda_k x)$$

$$= 2 \sum_{k=1}^{\infty} \frac{J_1(\lambda_k x)}{2\, J_1(\lambda_k) - \lambda_k\, J_0(\lambda_k)} , \qquad 0 < x < 1.$$

Reference to Table II,[9] which gives the roots of $J_1(x) = 0$ and maxima and minima of $J_0(x)$, gives

$$f(x) = \frac{2\, J_1(3.8317\, x)}{(3.8317)(0.4028)} - \frac{2\, J_1(7.0156\, x)}{(7.0156)(0.3001)}$$

$$+ \frac{2\, J_1(10.1735\, x)}{(10.1735)(0.2497)} - \frac{2\, J_1(13.3237\, x)}{(13.3237)(0.2184)} + \cdots ,$$

where

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3\, 2!} + \frac{x^5}{2^5 2!\, 3!} - \cdots + \frac{(-1)^k\, x^{2k+1}}{2^{2k+1}\, k!\, (k+1)!} ,$$

$$k = 0, 1, 2, \ldots .$$

A graphic representation of the first few terms of the above series is illustrated in Figure 14.

[9]Jahnke and Emde, op. cit., p. 166.

n	x_n	$J_0(x_n) = \begin{matrix}\text{Min}\\\text{Max}\end{matrix}$
1	3.8317	- 0.4028
2	7.0156	+ 0.3001
3	10.1735	- 0.2497
4	13.3237	+ 0.2184
5	16.4706	- 0.1965
6	19.6159	+ 0.1801
7	22.7601	- 0.1672
8	25.9037	+ 0.1567
9	29.0468	- 0.1480
10	32.1897	+ 0.1406

TABLE II

Roots of $J_1(x) = 0$ and maxima and minima of $J_0(x)$

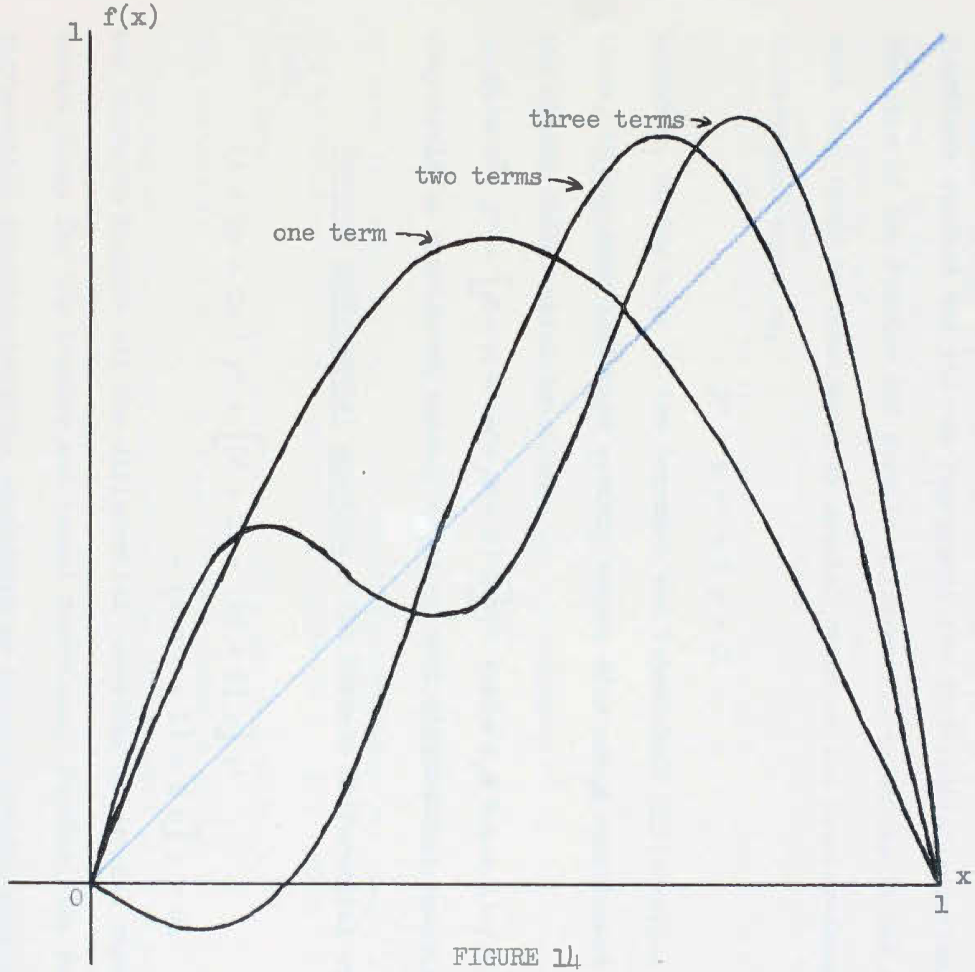

FIGURE 14

A Graphic Representation of the Bessel Series: $f(x) = x, \ 0 < x < 1.$

CHAPTER X

SUMMARY

Differential equations. It was seen that each of the orthogonal functions studied had its own representative differential equation. In the case of the Fourier and Hermite differential equations it was seen that they could be considered as special cases of the Fourier-Hermite differential equation,

$$y'' + A\, y' + B\, y = 0.$$

Likewise, in the case of the Legendre and Tchebicheff differential equations, it was seen that with certain values of α and β substituted into the Jacobi differential equation,

$$(1 - x^2)\, y'' + \left[\beta - \alpha - (\alpha + \beta + 2)\, x\right] y' + n(\alpha + \beta + n + 1)\, y = 0,$$

they could be considered special cases of that differential equation.

General differential equation. The Pearson differential equation,

$$(A + Bx + Cx)\, y'' + \left[(B + D) + (2C + E)\, x\right] y' - \left[C\, n(n + 1) + E\, n\right] y = 0,$$

was shown to include all the differential equations as special cases except those for the Fourier and Bessel functions. Further, the Pearson differential equation could be considered as being a special case of the more general differential equation

$$(A_1 + B_1 x\ C_1 x^2)\, y'' + (D_1 + E_1 x + F_1 x^2)\, y' + (G_1 + H_1 x + J_1 x^2)\, y = 0,$$

which included all of the differential equations associated with the orthogonal functions studied.

Sturm-Liouville theory. Even though the full theory embodied in the Sturm-Liouville theory was not applied to each of the differential equations, it was noted that the weight function and related parts could be obtained in each case. Thus, the ultimate establishment of the orthogonal functions and determination of the coefficients, when the functions were utilized in series form to represent a function of x, was expedited.

Determination of the coefficients. The coefficients of the orthogonal functions employed in a series representation of a function were found in a similar manner in each case studied. Basically each series was multiplied through by certain functions and integrated term by term over the interval of orthogonality. The property of orthogonality of the functions then reduced the series to a right-hand and a left-hand term. It was then possible to write the coefficient in terms of two integrals, which for the cases studied could be evaluated. The evaluation of these integrals, however, can be expected to be more difficult for some functions than for others.

Expansion of an arbitrary function in series. In each case studied two arbitrary functions were represented in a series form that utilized the various orthogonal functions. The arbitrary functions were kept the same in each case, where possible, in order to illustrate how the graphic representations were similar.

It was noted that in each graphic representation the plot of the first term plus the second term of the series came closer to the arbitrary function than did the first term by itself. The function was more nearly represented as more and more terms were added together. A partial exception to this, however, was noted in the case of the Laguerre series, where the first arbitrary function was represented by the first term of the series, and the second arbitrary function was represented by the first two terms. This fact can be attributed in part to a relationship between the arbitrary functions selected and the Laguerre series, rather than to an inherent quality of the Laguerre series for all functions.

[illegible]

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